

Approximate reversal of quantum Gaussian dynamics

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Abstract

Recently, there has been focus on determining the conditions under which the data processing inequality for quantum relative entropy is satisfied with approximate equality. The solution of the exact equality case is due to Petz, who showed that the quantum relative entropy between two quantum states stays the same after the action of a quantum channel if and only if there is a *reversal channel* that recovers the original states after the channel acts. Furthermore, this reversal channel can be constructed explicitly and is now called the *Petz recovery map*. Recent developments have shown that a variation of the Petz recovery map works well for recovery in the case of approximate equality of the data processing inequality. Our main contribution here is a proof that bosonic Gaussian states and channels possess a particular closure property, namely, that the Petz recovery map associated to a bosonic Gaussian state σ and a bosonic Gaussian channel \mathcal{N} is itself a bosonic Gaussian channel. We furthermore give an explicit construction of the Petz recovery map in this case, in terms of the mean vector and covariance matrix of the state σ and the Gaussian specification of the channel \mathcal{N} .

1 Introduction

1.1 Introduction to recoverability in quantum information

Strong subadditivity of quantum entropy is one of the cornerstones of quantum information theory, on which many fundamental results rely. Defining the conditional mutual information of a tripartite state ρ_{ABC} as

$$I(A; B|C)_\rho := S(AC)_\rho + S(BC)_\rho - S(ABC)_\rho - S(C)_\rho, \quad (1.1)$$

where $S(G)_\sigma \equiv -\text{Tr}[\sigma_G \log \sigma_G]$ is the quantum entropy of a state σ_G of a system G , strong subadditivity is equivalent to the non-negativity of conditional mutual information: $I(A; B|C)_\rho \geq 0$. Initially conjectured in 1967 [RR67, IR68], it was subsequently proven six years later [LR73a, LR73b]. Afterward, its equivalence to the data processing inequality for the quantum relative entropy [Ume62] was realized [Uhl73, Lin74, Lin75, Rus02]. This latter inequality has the form

$$D(\rho||\sigma) \geq D(\mathcal{N}(\rho)||\mathcal{N}(\sigma)), \quad (1.2)$$

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being valid for all states ρ, σ and all quantum channels \mathcal{N} (completely positive, trace-preserving maps). Here, the quantum relative entropy is defined for quantum states ρ and σ as

$$D(\rho\|\sigma) \equiv \text{Tr}[\rho(\log \rho - \log \sigma)], \quad (1.3)$$

whenever the support of ρ is contained in the support of σ , and it is set to $+\infty$ otherwise [Ume62].

The interest in strong subadditivity has not fallen over time, and many different proofs for it have been proposed in the last four decades (see for instance [NP05]). At the same time, new improvements of the original inequality have recently been found. Extending methods originally proposed in [Eff09], an operator generalization of strong subadditivity was recently proven in [Kim12].

A line of research which is of particular interest to us focuses on investigating the conditions under which strong subadditivity, or more generally the data processing inequality for relative entropy, is satisfied with equality or approximate equality. The solution of the exact equality case dates back to the 1980s: in [Pet86, Pet88, Pet03], it was shown that the relative entropy between two states stays the same after the action of a quantum channel if and only if there is a *recovery channel* bringing back both images to the original states. Furthermore, this reversing channel can be constructed explicitly and now takes the name *Petz recovery map*. Afterward, [MP04, Mos05] proved a structure theorem giving a form for states and a channel saturating the data-processing inequality for relative entropy, and, related to this development, the form of tripartite states satisfying strong subadditivity with equality was determined in [HJPW03].

Characterising the structure of states for which strong subadditivity is nearly saturated requires different techniques, and progress was not made until more recently. In 2011, a lower bound on conditional mutual information in terms of one-way LOCC norms [MWW09] was proven in [BCY11], the motivation for [BCY11] lying in the question of faithfulness of an entanglement measure called squashed entanglement [CW04] (see also [Tuc99, Tuc02] for discussions related to squashed entanglement). Later on, a conjecture put forward in [WL12] proposed another operationally meaningful remainder term for the relative entropy decrease induced by a quantum channel, given by the relative entropy between the state ρ and a “recovered version” of $\mathcal{N}(\rho)$. The authors of [WL12] proposed the following conjecture as a refinement of (1.2):

$$D(\rho\|\sigma) \stackrel{?}{\geq} D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) + D(\rho\|(\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N})(\rho)), \quad (1.4)$$

where $\mathcal{R}_{\sigma, \mathcal{N}}$ should be a quantum channel depending only on σ and \mathcal{N} and such that $(\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N})(\sigma) = \sigma$. The authors of [WL12] proved (1.4) in the classical case, when the states ρ and σ commute and the channel is classical as well, and they showed how the recovery channel in this case can be taken as the Petz recovery map. This conjecture has now been proven in a number of special, yet physically relevant cases as well [AWWW15, BDW16, ML16, LW16]. Unfortunately, the authors of [WL12] showed that in the general quantum case, $\mathcal{R}_{\sigma, \mathcal{N}}$ in (1.4) cannot be taken as the Petz recovery map. For further details, see also [Kim13, LW14], and for related conjectures, see [BSW15a, SBW15].

While the general form of the conjecture in (1.4) remains unproven, in [FR15], it was shown that if the conditional mutual information $I(A; B|C)_\rho$ is small, then the state ρ_{ABC} can be very well approximated by one of its “reconstructed” versions $\mathcal{R}_{C \rightarrow BC}(\rho_{AC})$. That is, the authors of [FR15] proved the following inequality:

$$I(A; B|C)_\rho \geq -\log F(\rho_{ABC}, \mathcal{R}_{C \rightarrow BC}(\rho_{AC})), \quad (1.5)$$

where F denotes the quantum fidelity [Uhl76], defined as $F(\omega, \tau) := \|\sqrt{\omega}\sqrt{\tau}\|_1^2$ for quantum states ω and τ , and $\mathcal{R}_{C \rightarrow BC}$ is a recovery channel taking an input system C to output systems BC .

Furthermore, the channel $\mathcal{R}_{C \rightarrow BC}$ can be taken as the Petz recovery map up to some unitary rotations preceding and following its action, but note that the unitary rotations given in [FR15] generally depend on the full state ρ_{ABC} .

After the result of [FR15] appeared, much activity surrounding entropy inequalities and recovery channels occurred. An alternative and simpler proof of the faithfulness of squashed entanglement following the lines of [WL12] immediately appeared [LW14], while an alternative proof of (1.5) that makes use of quantum state redistribution [DY08, YD09] appeared in [BHOS14]. In [SFR16], an important particular case of (1.5) was proven; that is, it was shown that the recovery map in (1.5) can be chosen to depend only on ρ_{BC} and to obey $\mathcal{R}_{C \rightarrow BC}(\rho_C) = \rho_{BC}$. A different approach was delivered in [Wil15], based on the methods of complex interpolation [BL76] and generalized Rényi entropies [BSW15a, SBW15]. The main result of [Wil15] states that a lower bound on the decrease in relative entropy induced by a quantum channel is given by the negative logarithm of the fidelity between the first state and its recovered version, which is a step closer to the proof of the conjecture in (1.4). However, the recovery term in [Wil15] is weaker than the right-hand side of (1.4), and the map appearing in it lacks one of the two properties that it is required to obey. Another step toward the proof of the conjecture in (1.4) was performed in [JRS⁺15], where a more general tool from complex analysis [Hir52] and the methods of [BSW15a, SBW15, Wil15] were exploited in order to prove a statement similar to (1.4), with the relative entropy on the right-hand side substituted by a negative log-fidelity, but with the recovery map depending only on σ and \mathcal{N} and furthermore satisfying $\mathcal{R}_{\sigma, \mathcal{N}}(\mathcal{N}(\sigma)) = \sigma$. Meanwhile, a systematic method for deriving matrix inequalities by forcing the operators to commute via the application of suitably chosen “pinching maps” was proposed in [SBT16]. This method as well as the complex interpolation techniques in [DW16] can be also applied to prove multioperator trace inequalities [DW16, SBT16, Wil16], which generalise the celebrated Golden-Thompson inequality $\text{Tr}[e^{X+Y}] \leq \text{Tr}[e^X e^Y]$ (X, Y hermitian) and the stronger statements given in [Lie73]. The results of [SBT16] also marked further progress toward establishing the conjecture in (1.4).

1.2 Introduction to quantum Gaussian states and channels

A major platform for the application of quantum information theory to physical information processing is constituted by quantum optics [GK04] with a finite number of electromagnetic modes or quantum harmonic oscillators. From the mathematical perspective, this framework can be thought of as quantum mechanics applied to separable Hilbert spaces endowed with a finite number of operators obeying canonical commutation relations [Ser17].

A typical free Hamiltonian of such a system is quadratic in the canonical operators, and in fact, a special role within this context is played by ground or thermal states of such Hamiltonians, commonly called *Gaussian states*. These states define a useful operational framework for several reasons, stemming from both physics and mathematics [ARL14, Ser17]. From the physical point of view, they are easily produced and manipulated in the laboratory and can be used to implement effective quantum protocols [BR04, WHTH07]. Mathematically convenient properties that qualify them as defining a legitimate framework include

1. the closure under so-called Gaussian unitary evolutions, that is, unitaries induced by piecewise time evolution via quadratic Hamiltonians, as well as more generally
2. the closure under Gaussian channels, which can be understood as the operation of adding an

ancillary system in a vacuum state, applying a global Gaussian unitary, and tracing out one of the subsystems [CEGH08].

Recently, more advanced “closure” properties have been established, such as the optimality of Gaussian states for optimising the output entropy of one-mode, phase-covariant quantum channels, even when a fixed value of the input entropy is prescribed [GHGP15, PTG17, PTG16a, PTG16b]. These facts have the striking implication that it suffices to select coding strategies according to Gaussian states in order to achieve optimal rates in several quantum communication tasks [GGL⁺04, WHG12, GGPCH14, QW17, WQ16, PTG16b].

1.3 Summary of main result

The main contribution of our paper is a proof that Gaussian states and channels possess another closure property: the Petz recovery map associated to a Gaussian state σ and a Gaussian channel \mathcal{N} is itself a Gaussian channel (see Theorem 1). Additionally, we achieve this result through an explicit construction of the action of such a Gaussian Petz channel, which lends itself to multiple applications. For instance, with the formulas we provide, it is possible to construct a counterexample to the inequality in (1.4), in which all the states and channels involved are Gaussian and $\mathcal{R}_{\sigma, \mathcal{N}}$ is the Petz recovery map. This is similar to what happens in the finite-dimensional case. Another application of our main result is a more explicit form for an entropy inequality from [JRS⁺15], whenever the states and channel involved are Gaussian.

More broadly, our result has implications for a resource theory of non-Gaussianity [BS02a, BS02b, BESP03, BvL05, Gou16], which is not currently complete but for which there has been notable progress. In particular, in such a theory, one takes the free states and free operations to be quantum Gaussian states and channels, respectively, and the expensive or resourceful ones to be non-Gaussian. Such an approach is motivated by concerns from quantum computation using continuous variables, in which universal quantum computation is enabled only when non-Gaussian operations are available [BS02a, BS02b, BSBN02], or from quantum communication theory, in which non-Gaussian operations are needed for quantum error correction [NFC09], for enhancements over classical communication strategies [TG14, LJP16], for discrimination of coherent states [TS08], or for effective quantum repeaters in quantum key distribution [NGGL14]. One might expect the Petz recovery channel to play a critical role in a resource theory of non-Gaussianity as it has in other resource theories [AWWW15, ML16, LW16]. As such, our result shows that, in such a resource theory, the Petz recovery channel is a free operation if the state σ is free and the forward channel \mathcal{N} is free as well. One can quantify non-Gaussianity of a quantum state ρ via the following information measure, known as the relative entropy of non-Gaussianity [GPB08, GP10]:

$$D_G(\rho) \equiv \min_{\sigma \in \mathcal{G}} D(\rho \| \sigma) = D(\rho \| \rho_G), \quad (1.6)$$

where \mathcal{G} denotes the set of Gaussian states and ρ_G denotes a quantum Gaussian state with the same mean vector and covariance matrix as ρ (that ρ_G is indeed the minimizer was proven in [MM13]). The relative entropy of non-Gaussianity has not been established as an operationally meaningful quantifier in the resource-theoretic sense, but one might think it to be the case in light of the prominence of relative-entropy quantifiers in other resource theories [BaG15]. However, if it eventually is, our work combined with the main result of [JRS⁺15] would be relevant, given that these results establish the following interesting inequality, holding for an arbitrary quantum state

ρ and quantum Gaussian channel \mathcal{N}_G :

$$D_G(\rho) \geq D_G(\mathcal{N}_G(\rho)) - \int_{\mathbb{R}} dt p(t) \log F(\rho, (\mathcal{P}_{\rho_G, \mathcal{N}_G}^{t/2} \circ \mathcal{N}_G)(\rho)), \quad (1.7)$$

where $p(t) := \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$ is a probability distribution parametrized by $t \in \mathbb{R}$ and $\mathcal{P}_{\rho_G, \mathcal{N}_G}^t$ is a rotated Petz channel [Wil15]. A corollary of our main result is that $\mathcal{P}_{\rho_G, \mathcal{N}_G}^t$ is a quantum Gaussian channel (Corollary 2). The inequality in (1.7) has an interpretation similar to that in previous works: if the relative entropy of non-Gaussianity does not decrease too much under the action of a free operation \mathcal{N}_G (so that $D_G(\rho) \approx D(\mathcal{N}_G(\rho))$), then one can approximately reverse the action of \mathcal{N}_G by employing a free operation $\mathcal{P}_{\rho_G, \mathcal{N}_G}^{t/2}$ chosen randomly according to $p(t)$. Note that one can also write the inequality above as follows:

$$D(\rho \| \rho_G) \geq D(\mathcal{N}_G(\rho) \| \mathcal{N}_G(\rho_G)) - \int_{\mathbb{R}} dt p(t) \log F(\rho, (\mathcal{P}_{\rho_G, \mathcal{N}_G}^{t/2} \circ \mathcal{N}_G)(\rho)). \quad (1.8)$$

We should note that the inequalities stated above are not in contradiction with the well known no-go theorem for Gaussian quantum error correction [NFC09]. The main result of [NFC09] is the following statement: if one is trying to use a Gaussian quantum channel to distill entanglement between spatially separated parties, then Gaussian encodings combined with Gaussian decodings are not helpful for this task, whenever performance is measured with respect to an entanglement measure called logarithmic negativity. In the inequalities in (1.7)–(1.8), the recovery channel is indeed a quantum Gaussian channel, but the only statement that these inequalities make is that the performance of the Gaussian Petz recovery channel for recovery is limited by the relative entropy difference $D(\rho \| \rho_G) - D(\mathcal{N}_G(\rho) \| \mathcal{N}_G(\rho_G))$.

Finally, we suspect that our main result about Gaussian Petz channels might be useful in contexts beyond the traditional ones in quantum information theory. Indeed, Petz recovery maps have recently been employed in the context of high-energy physics, quantum many-body physics, and topological order [SM16, ZS16, Kim16], and so our result here could be useful if the states involved in those contexts are Gaussian states.

This paper is structured as follows. In Section 2, we review some background material and establish notation. In particular, we review the Petz recovery map (Section 2.1) and bosonic Gaussian states and channels (Section 2.2). In Section 3, we state our main result, Theorem 1, which establishes that the Petz recovery map for a Gaussian state σ and a Gaussian channel \mathcal{N} is itself a Gaussian channel, and we give an explicit form for it in terms of the parameters that characterize σ and \mathcal{N} . Corollary 2 establishes a similar result for the rotated Petz maps from [Wil15]. Our proof of Theorem 1 is divided into four parts, given in Sections 3.1–3.4. We conclude in Section 4 with a summary and some open questions. We point the interested reader to Appendix A, in which we give a method for computing products of exponentials of inhomogeneous quadratic Hamiltonians, building upon [BB69]. Although results of [Pet86, Pet88, OP93] establish that the Petz map is completely positive and trace-preserving, Appendix B offers a different argument that the Gaussian Petz map is completely positive.

2 Background and notation

2.1 Petz recovery map

As discussed in Section 1.1, the Petz recovery map is a notable object playing a crucial role in the theory of quantum recoverability. It has been interpreted in [LS13] as a quantum generalization of the Bayes rule from probability theory. Given a state σ and a channel \mathcal{N} , the associated Petz map $\mathcal{P}_{\sigma, \mathcal{N}}$ is defined as a linear map satisfying the following [Pet86, Pet88, OP93]:

$$\langle A, \mathcal{N}^\dagger(B) \rangle_\sigma = \langle \mathcal{P}_{\sigma, \mathcal{N}}^\dagger(A), B \rangle_{\mathcal{N}(\sigma)}, \quad \forall A, B, \quad (2.1)$$

where A and B are bounded operators and the weighted Hilbert–Schmidt inner product is defined for bounded operators τ_1 and τ_2 and a trace-class operator ξ as

$$\langle \tau_1, \tau_2 \rangle_\xi \equiv \text{Tr}[\tau_1^\dagger \xi^{1/2} \tau_2 \xi^{1/2}]. \quad (2.2)$$

The map $\mathcal{P}_{\sigma, \mathcal{N}}$ is unique if $\mathcal{N}(\sigma)$ is a faithful operator [Pet86, Pet88, OP93], and otherwise, it is unique on the support of this operator. If σ acts on a finite-dimensional Hilbert space and \mathcal{N} is a quantum channel with finite-dimensional inputs and outputs, then the Petz map takes the following explicit form [HJPW03]:

$$\mathcal{P}_{\sigma, \mathcal{N}}(\omega) \equiv \sigma^{1/2} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-1/2} \omega \mathcal{N}(\sigma)^{-1/2} \right) \sigma^{1/2}, \quad (2.3)$$

where $\mathcal{N}(\sigma)^{-1/2}$ is understood as a generalized inverse (i.e., inverse on the support of $\mathcal{N}(\sigma)$). Sometimes we omit the dependence of \mathcal{P} on σ and \mathcal{N} for the sake of simplicity. A rotated Petz map $\mathcal{P}_{\sigma, \mathcal{N}}^t$ for $t \in \mathbb{R}$, a state σ , and a channel \mathcal{N} is defined as [Wil15]

$$\mathcal{P}_{\sigma, \mathcal{N}}^t(\omega) \equiv \sigma^{it} \mathcal{P}_{\sigma, \mathcal{N}}(\mathcal{N}(\sigma)^{-it} \omega \mathcal{N}(\sigma)^{it}) \sigma^{-it}, \quad (2.4)$$

with $\sigma^{it} = \exp(it \log \sigma)$ being understood as a unitary evolution according to the Hamiltonian $\log \sigma$.

2.2 Quantum Gaussian states and channels

Here we provide some background on quantum Gaussian states and channels (see [CEGH08, ARL14, Ser17] for reviews). An n -mode quantum system is described by a density operator acting on a tensor-product Hilbert space. To the j th Hilbert space in the tensor product, for $j \in \{1, \dots, n\}$, we let x_j and p_j denote the position- and momentum-quadrature operator, respectively. These operators satisfy the canonical commutation relations: $[x_j, p_k] = i\delta_{j,k}$, where we have set $\hbar = 1$. It is convenient to form a vector $r = (x_1, \dots, x_n, p_1, \dots, p_n)^T$ from these operators, and then we can rewrite the canonical commutation relations in matrix form as follows:

$$[r, r^T] = i\Omega, \quad (2.5)$$

where

$$\Omega \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_n, \quad (2.6)$$

and I_n denotes the $n \times n$ identity matrix. We often make use of the identities $\Omega^T \Omega = I$ and $\Omega^T = -\Omega$.

The displacement (Weyl) operator D_z plays an important role in Gaussian quantum information, defined for $z \in \mathbb{R}^{2n}$ as

$$D_z \equiv \exp(iz^T \Omega r). \quad (2.7)$$

For $z_1, z_2 \in \mathbb{R}^{2n}$, the displacement operators satisfy the following composition rule:

$$D_{z_1} D_{z_2} = D_{z_1+z_2} e^{-\frac{i}{2} z_1^T \Omega z_2}. \quad (2.8)$$

It can be shown that displacement operators form a complete, orthogonal set of operators, and their Hilbert–Schmidt orthogonality relation is as follows:

$$\text{Tr}[D_{z_1} D_{-z_2}] = (2\pi)^n \delta(z_1 - z_2). \quad (2.9)$$

Moreover, due to their completeness, these operators allow for a Fourier-Weyl expansion of a quantum state, in terms of a characteristic function. In more detail, a quantum state ρ has a characteristic function $\chi_\rho(w)$, defined as

$$\chi_\rho(w) \equiv \text{Tr}[\rho D_{-w}], \quad (2.10)$$

and the original state ρ can be written in terms of $\chi_\rho(w)$ as

$$\rho = \int \frac{d^{2n}w}{(2\pi)^n} \chi_\rho(w) D_w. \quad (2.11)$$

The mean vector $s_\rho \in \mathbb{R}^{2n}$ and $2n \times 2n$ covariance matrix V_ρ of a quantum state ρ are defined as

$$s_\rho \equiv \langle r \rangle_\rho = \text{Tr}[r\rho], \quad (2.12)$$

$$V_\rho \equiv \langle \{r - s_\rho, r^T - s_\rho^T\} \rangle_\rho = \text{Tr}[\{r - s_\rho, r^T - s_\rho^T\} \rho]. \quad (2.13)$$

It follows from the above definition that the covariance matrix V_ρ is symmetric.

A quantum Gaussian state is a ground or thermal state of a Hamiltonian that is quadratic in the position- and momentum-quadrature operators. In particular, up to an irrelevant additive constant, any such Hamiltonian has the form $\frac{1}{2} (r - s)^T H (r - s)$, where $s \in \mathbb{R}^{2n}$ and H is a $2n \times 2n$ positive definite matrix that we refer to as the Hamiltonian matrix. Then a quantum Gaussian state ρ takes the form

$$\rho = Z_\rho^{-1} \exp\left(-\frac{1}{2} (r - s_\rho)^T H_\rho (r - s_\rho)\right), \quad (2.14)$$

where $Z_\rho \equiv \text{Tr}[\exp(-\frac{1}{2} (r - s_\rho)^T H_\rho (r - s_\rho))]$ and one can show that $\langle r \rangle_\rho = s_\rho \in \mathbb{R}^{2n}$ (i.e., s_ρ is the mean vector of ρ). Defining

$$V_\rho \equiv \coth\left(\frac{i\Omega H_\rho}{2}\right) i\Omega, \quad (2.15)$$

one can also show that V_ρ is the covariance matrix of ρ , whose matrix elements satisfy $V_\rho^{j,k} = \langle \{r_j - s_\rho^j, r_k - s_\rho^k\} \rangle_\rho$ and the Heisenberg uncertainty relation [SMD94]:

$$V_\rho + i\Omega \geq 0. \quad (2.16)$$

A quantum Gaussian state is faithful (having full support) if $V_\rho + i\Omega > 0$.

A quantum Gaussian state ρ with mean vector s_ρ and covariance matrix V_ρ has the following Gaussian characteristic function:

$$\chi_\rho(w) = \exp\left(-\frac{1}{4}(\Omega w)^T V_\rho \Omega w + i(\Omega w)^T s_\rho\right), \quad (2.17)$$

so that it can be written in the following way:

$$\rho = \int \frac{d^{2n}w}{(2\pi)^n} \exp\left(-\frac{1}{4}(\Omega w)^T V_\rho \Omega w + i(\Omega w)^T s_\rho\right) D_w. \quad (2.18)$$

After a change of variables ($w \rightarrow \Omega w$), this representation becomes

$$\rho = \int \frac{d^{2n}w}{(2\pi)^n} \exp\left(-\frac{1}{4}w^T V_\rho w - iw^T s_\rho\right) D_{\Omega w}. \quad (2.19)$$

A quantum Gaussian channel is a completely positive, trace-preserving map that takes Gaussian input states to Gaussian output states. A quantum Gaussian channel \mathcal{N} that takes n -mode Gaussian input states to m -mode Gaussian output states is specified by a $2m \times 2n$ transformation matrix X , a $2m \times 2m$ positive semi-definite, additive noise matrix Y , and a displacement vector $\delta \in \mathbb{R}^{2n}$. The action of such a channel on a generic state ρ with characteristic function $\chi_\rho(w)$ is to output a state $\mathcal{N}(\rho)$ having the following characteristic function:

$$\chi_{\mathcal{N}(\rho)}(w) = \chi_\rho(\Omega^T X^T \Omega w) \exp\left(-\frac{1}{4}(\Omega w)^T Y \Omega w + i(\Omega w)^T \delta\right). \quad (2.20)$$

Then the channel \mathcal{N} leads to the following transformation of the covariance matrix V and mean vector s of an input quantum Gaussian state:

$$\mathcal{N}: \begin{cases} V & \mapsto XVX^T + Y \\ s & \mapsto Xs + \delta \end{cases}. \quad (2.21)$$

The matrices X and Y should satisfy the following condition in order for the map \mathcal{N} to be completely positive:

$$Y + i\Omega \geq iX\Omega X^T. \quad (2.22)$$

The adjoint of a quantum channel \mathcal{N} is defined as the unique linear map satisfying the following for all A and B :

$$\langle A, \mathcal{N}(B) \rangle = \langle \mathcal{N}^\dagger(A), B \rangle, \quad (2.23)$$

where B is an arbitrary trace-class operator, A is an arbitrary bounded operator, and the Hilbert–Schmidt inner product is defined for operators A_1 and A_2 as $\langle A_1, A_2 \rangle \equiv \text{Tr}[A_1^\dagger A_2]$. The adjoint map \mathcal{N}^\dagger is completely positive and unital if \mathcal{N} is completely positive and trace-preserving. The action of the adjoint \mathcal{N}^\dagger of a quantum Gaussian channel \mathcal{N} defined by (2.21) is as follows [CEGH08, GLS16], when acting on a displacement operator $D_{\Omega z}$:

$$\mathcal{N}^\dagger(D_{\Omega z}) = D_{\Omega X^T z} \exp\left(-\frac{1}{4}z^T Y z + iz^T \delta\right). \quad (2.24)$$

The action of the adjoint \mathcal{N}^\dagger on a quantum Gaussian state with covariance matrix V and mean vector s is then to output a quantum Gaussian operator described by covariance matrix $X^{-1}(V + Y)X^{-T}$

and mean vector $X^{-1}(s - \delta)$ whenever X is invertible [GLS16, Appendix B]. We summarize these transformation rules as follows:

$$\mathcal{N}^\dagger : \begin{cases} V & \mapsto X^{-1}(V + Y)X^{-T} \\ s & \mapsto X^{-1}(s - \delta) \end{cases}. \quad (2.25)$$

Typically one thinks of the channel \mathcal{N} as acting in the Schrödinger picture, taking input states to output states, and one thinks of the adjoint \mathcal{N}^\dagger as acting in the Heisenberg picture, taking input bounded operators to output bounded operators. So this is why we have specified the channel \mathcal{N} in terms of its action on characteristic functions, which describe states, and the adjoint \mathcal{N}^\dagger in terms of its action on displacement operators, a natural choice of bounded operators in our context here.

Often we find it useful to write

$$\sigma = D_{s_\sigma}^\dagger \sigma_0 D_{s_\sigma}, \quad (2.26)$$

where σ_0 is a Gaussian state with the same covariance matrix as σ but with vanishing mean vector. Analogously, the channel \mathcal{N} in (2.20) admits the following decomposition:

$$\mathcal{N}(\cdot) = D_\delta^\dagger \mathcal{N}_0(\cdot) D_\delta, \quad (2.27)$$

where \mathcal{N}_0 is a zero-displacement Gaussian channel, acting as in (2.21) but with $\delta = 0$. Taking the adjoint gives

$$\mathcal{N}^\dagger(\cdot) = \mathcal{N}_0^\dagger(D_\delta(\cdot)D_\delta^\dagger). \quad (2.28)$$

Applying \mathcal{N} to σ yields

$$\mathcal{N}(\sigma) = D_{Xs+\delta}^\dagger \mathcal{N}_0(\sigma_0) D_{Xs+\delta}, \quad (2.29)$$

which follows from (2.21). We also make use of the following channel covariance relations:

$$\mathcal{N}(D_\gamma^\dagger(\cdot)D_\gamma) = D_{X\gamma+\delta}^\dagger \mathcal{N}_0(\cdot) D_{X\gamma+\delta}, \quad (2.30)$$

$$\mathcal{N}^\dagger(D_\gamma^\dagger(\cdot)D_\gamma) = D_{X^{-1}(\gamma-\delta)}^\dagger \mathcal{N}_0^\dagger(\cdot) D_{X^{-1}(\gamma-\delta)}, \quad (2.31)$$

which follow from (2.20), (2.21), (2.24), and (2.25). Note that (2.31) holds whenever X is invertible.

Finally, given a Gaussian state σ with mean vector s_σ and covariance matrix V_σ , we can consider a unitary rotation of the form $\sigma^{it} = \exp(it \log \sigma)$ for $t \in \mathbb{R}$. By using the representation in (2.14) with the Hamiltonian matrix H_σ , we can write the unitary σ^{it} as

$$\sigma^{it} = \exp\left(-\frac{i}{2}(r - s_\sigma)^T H_\sigma t (r - s_\sigma)\right) \exp(-it \log Z_\sigma) \quad (2.32)$$

$$= D_{-s_\sigma} \left[\exp\left(\frac{i}{2}r^T (-H_\sigma t) r\right) \exp(-it \log Z_\sigma) \right] D_{s_\sigma}, \quad (2.33)$$

where we have used the fact that $(r - s_\sigma)^T H_\sigma (r - s_\sigma) = D_{-s_\sigma} r^T H_\sigma r D_{s_\sigma}$ and the operator identity $B \exp(A) B^{-1} = \exp(BAB^{-1})$. The unitary σ^{it} is a Gaussian unitary because it is generated by a Hamiltonian no more than quadratic in the position- and momentum-quadrature operators. Let us define the symplectic transformation corresponding to the unitary $\exp(\frac{i}{2}r^T (-H_\sigma t) r)$ as

$$S_{\sigma,t} \equiv \exp(\Omega H_\sigma t), \quad (2.34)$$

so that

$$\sigma^{it} r \sigma^{-it} = S_{\sigma, -t} (r - s_\sigma) + s_\sigma, \quad (2.35)$$

where we used that $D_{s_\sigma} r D_{-s_\sigma} = r + s_\sigma$. The above formula implies that

$$V_{\sigma^{it} \omega \sigma^{-it}} = S_{\sigma, t} V_\omega S_{\sigma, t}^T, \quad (2.36)$$

$$s_{\sigma^{it} \omega \sigma^{-it}} = S_{\sigma, t} (s_\rho - s_\sigma) + s_\sigma. \quad (2.37)$$

3 Main result: Petz map as a quantum Gaussian channel

Our main result is the following theorem:

Theorem 1 *Let σ be a quantum Gaussian state with mean vector s_σ and covariance matrix V_σ , and let \mathcal{N} be a quantum Gaussian channel with its action on an input state as described in (2.21). Suppose furthermore that $\mathcal{N}(\sigma)$ is a faithful quantum state. Then the Petz recovery map $\mathcal{P}_{\sigma, \mathcal{N}}$ is a quantum Gaussian channel with the following action:*

$$\mathcal{P}_{\sigma, \mathcal{N}} : \begin{cases} V & \mapsto X_P V X_P^T + Y_P \\ s & \mapsto X_P s + \delta_P \end{cases}, \quad (3.1)$$

where

$$X_P \equiv \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma X^T \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}}^{-1} V_{\mathcal{N}(\sigma)}^{-1}, \quad (3.2)$$

$$Y_P \equiv V_\sigma - X_P V_{\mathcal{N}(\sigma)} X_P^T, \quad (3.3)$$

$$\delta_P \equiv s_\sigma - X_P (X s_\sigma + \delta), \quad (3.4)$$

$$V_{\mathcal{N}(\sigma)} = X V_\sigma X^T + Y. \quad (3.5)$$

That is, $\mathcal{P}_{\sigma, \mathcal{N}}$ in (3.1) is the unique linear map satisfying (2.1) for σ and \mathcal{N} as described above.

The following corollary is a direct consequence of Theorem 1 and the discussion surrounding (2.32)–(2.35):

Corollary 2 *For σ and \mathcal{N} as given in Theorem 1, the rotated Petz map $\mathcal{P}_{\sigma, \mathcal{N}}^t$ (defined in (2.4)) is also a quantum Gaussian channel with the same action as the Petz recovery channel $\mathcal{P}_{\sigma, \mathcal{N}}$ but with the substitutions*

$$X_P \rightarrow X_P^t \equiv S_{\sigma, t} X_P S_{\mathcal{N}(\sigma), -t}, \quad (3.6)$$

$$Y_P \rightarrow Y_P^t \equiv S_{\sigma, t} Y_P S_{\sigma, t}^T, \quad (3.7)$$

$$\delta_P \rightarrow \delta_P^t \equiv s_\sigma - X_P^t (X s_\sigma + \delta). \quad (3.8)$$

That is, $\mathcal{P}_{\sigma, \mathcal{N}}^t$ is a quantum Gaussian channel with the following action:

$$\mathcal{P}_{\sigma, \mathcal{N}}^t : \begin{cases} V & \mapsto X_P^t V (X_P^t)^T + Y_P^t \\ s & \mapsto X_P^t s + \delta_P^t \end{cases}. \quad (3.9)$$

Remark 3 The following entropy inequality was proven to hold whenever ρ and σ are density operators and \mathcal{N} is a quantum channel [JRS⁺15]:

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) - \int_{\mathbb{R}} dt \, p(t) \log F(\rho, (\mathcal{P}_{\sigma, \mathcal{N}}^{t/2} \circ \mathcal{N})(\rho)), \quad (3.10)$$

where $p(t) := \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$ is a probability distribution parametrized by $t \in \mathbb{R}$. In the case that ρ and σ are quantum Gaussian states and \mathcal{N} is a quantum Gaussian channel, Corollary 2 allows us to conclude that $\mathcal{P}_{\sigma, \mathcal{N}}^{t/2}$ is a quantum Gaussian channel for all $t \in \mathbb{R}$. Furthermore, there are explicit, compact formulas for the relative entropy [SW01, Che05, PLOB16] and fidelity [BBP15] of two quantum Gaussian states. In both cases, the formulas are given exclusively in terms of the mean vectors and covariance matrices of the involved states. Thus, when the states and channel involved are all Gaussian, the above inequality can be rewritten in a simpler form involving only finite-dimensional matrices instead of trace-class operators acting on infinite-dimensional Hilbert spaces.

The forthcoming subsections establish a proof of Theorem 1. Before delving into our proof, we highlight our proof strategy, which proceeds according to the following steps:

1. Even though the explicit form of the Petz map in (2.3) is not generally valid in the infinite-dimensional case because the inverse of a density operator may be unbounded, we work with it anyway, as an ansatz (call this **Ansatz 1**). Under Ansatz 1, we first show that it suffices to consider the case when the state σ is a zero-mean Gaussian state and the channel \mathcal{N} does not apply any displacement to the mean vector of its input, so that $s_\sigma = 0$ and $\delta = 0$, with δ defined in (2.20) and (2.21).
2. Under the same Ansatz 1, we arrive at the hypothesis that (3.1) gives the explicit form for the action of the Petz map on a Gaussian input state. Recall from (2.3) that the Petz map is a serial concatenation of three completely positive maps:

$$(\cdot) \rightarrow \mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}, \quad (3.11)$$

$$(\cdot) \rightarrow \mathcal{N}^\dagger(\cdot), \quad (3.12)$$

$$(\cdot) \rightarrow \sigma^{1/2}(\cdot)\sigma^{1/2}. \quad (3.13)$$

To handle the first completely positive map in (3.11), we proceed with an additional ansatz (**Ansatz 2**) that taking the inverse of a Gaussian state corresponds to negating its covariance matrix. This is motivated by the representation in (2.14), in which inverting the density operator has the effect of negating the Hamiltonian matrix, which in turn has the effect of negating the covariance matrix due to the fact that $\operatorname{arccoth}$ is an odd function. Furthermore, results of [BBP15, Appendix B-2] allow us to conclude that sandwiching a Gaussian state by the square root of another Gaussian state is a Gaussian map resulting in another unnormalized, Gaussian state. To handle the second map in (3.12), we can directly apply a result given in [GLS16, Appendix B], which gives an explicit form for the action of the adjoint of a Gaussian channel on a Gaussian state (see also the review in (2.25)). We also work with a final **Ansatz 3**, which is the assumption that the matrix X in (2.21) is invertible. Later, we show how this assumption is not necessary. To handle the third completely positive map in (3.13), we again apply the aforementioned result about sandwiching a Gaussian state by the square root of another.

3. After arriving at an explicit form for the Petz map by using Ansatzes 1–3, we verify that this explicit form satisfies the equations in (2.1) whenever the operators A and B are Hilbert–Schmidt operators.
4. We finally employ a limiting argument to conclude that if (2.1) is satisfied when A and B are Hilbert–Schmidt operators, then the equations are satisfied when A and B are arbitrary bounded operators. By a result of [Pet86, Pet88, OP93], we can finally conclude that the Gaussian channel given in Theorem 1 is the unique quantum channel satisfying (2.1). This step then concludes our proof of Theorem 1.

In the subsections that follow, we give detailed proofs for each step above.

3.1 Step 1: Sufficiency of focusing on zero-mean Gaussian states and zero-displacement Gaussian channels

As mentioned above, we employ Ansatz 1 in this first step, in which we work with the explicit form of the Petz map in (2.3), in spite of the fact that the inverse of a Gaussian density operator is unbounded. Let σ be a quantum Gaussian state with mean vector s_σ and covariance matrix V_σ , and let \mathcal{N} be a quantum Gaussian channel with the action on an input state as described in (2.21).

In this first step, we show how it suffices to consider the case $s_\sigma = \delta = 0$ in (2.3). To see this, consider the action of the Petz map $\mathcal{P}_{\sigma, \mathcal{N}}$ on an arbitrary input state ω :

$$\mathcal{P}_{\sigma, \mathcal{N}}(\omega) = \sigma^{1/2} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-1/2} \omega \mathcal{N}(\sigma)^{-1/2} \right) \sigma^{1/2} \quad (3.14)$$

$$= \left(D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} \right) \mathcal{N}_0^\dagger \left[D_\delta D_{X_{s_\sigma+\delta}}^\dagger \mathcal{N}_0(\sigma_0)^{-1/2} D_{X_{s_\sigma+\delta}} \omega D_{X_{s_\sigma+\delta}}^\dagger \mathcal{N}_0(\sigma_0)^{-1/2} D_{X_{s_\sigma+\delta}} D_\delta^\dagger \right] \\ \times \left(D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} \right) \quad (3.15)$$

$$= \left(D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} \right) \mathcal{N}_0^\dagger \left[D_{X_{s_\sigma}}^\dagger \mathcal{N}_0(\sigma_0)^{-1/2} D_{X_{s_\sigma+\delta}} \omega D_{X_{s_\sigma+\delta}}^\dagger \mathcal{N}_0(\sigma_0)^{-1/2} D_{X_{s_\sigma}} \right] \\ \times \left(D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} \right) \quad (3.16)$$

$$= D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} D_{X^{-1}(X_{s_\sigma})}^\dagger \mathcal{N}_0^\dagger \left[\mathcal{N}_0(\sigma_0)^{-1/2} D_{X_{s_\sigma+\delta}} \omega D_{X_{s_\sigma+\delta}}^\dagger \mathcal{N}_0(\sigma_0)^{-1/2} \right] \\ \times D_{X^{-1}(X_{s_\sigma})} D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} \quad (3.17)$$

$$= D_{s_\sigma}^\dagger \sigma_0^{1/2} \mathcal{N}_0^\dagger \left[\mathcal{N}_0(\sigma_0)^{-1/2} D_{X_{s_\sigma+\delta}} \omega D_{X_{s_\sigma+\delta}}^\dagger \mathcal{N}_0(\sigma_0)^{-1/2} \right] \sigma_0^{1/2} D_{s_\sigma} \quad (3.18)$$

$$= D_{s_\sigma}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0} \left(D_{X_{s_\sigma+\delta}} \omega D_{X_{s_\sigma+\delta}}^\dagger \right) D_{s_\sigma}. \quad (3.19)$$

For the first equality, we use the definition of the Petz map and Ansatz 1. The second equality follows from (2.26)–(2.29) and the fact that $f(UAU^\dagger) = Uf(A)U^\dagger$ for a function f , a unitary operator U , and a Hermitian operator A . The third equality follows because $D_\delta D_{X_{s_\sigma+\delta}}^\dagger = D_{X_{s_\sigma}}^\dagger e^{i\phi}$ for ϕ a phase. The fourth equality follows from the adjoint channel covariance relation in (2.31) and Ansatz 3. The fifth equality follows because $D_{s_\sigma} D_{X^{-1}(X_{s_\sigma})}^\dagger = e^{i\varphi} I$ for some phase φ . The final equality follows by recognizing the form of the Petz map $\mathcal{P}_{\sigma_0, \mathcal{N}_0}$, corresponding to the zero-mean state σ_0 and the zero-displacement channel \mathcal{N}_0 .

The above reasoning suggests that we should focus on determining an explicit form for $\mathcal{P}_{\sigma_0, \mathcal{N}_0}(\omega)$. That is, the above reasoning suggests that an arbitrary Petz map $\mathcal{P}_{\sigma, \mathcal{N}}$ can be realized as a serial

concatenation of the displacement $D_{X_{s_\sigma} + \delta}$, the Petz map $\mathcal{P}_{\sigma_0, \mathcal{N}_0}$, and the displacement $D_{s_\sigma}^\dagger$. After we give an explicit form for $\mathcal{P}_{\sigma_0, \mathcal{N}_0}$ as a quantum Gaussian channel with matrices X_P and Y_P , it should become clear why the displacement δ_P in the Petz map $\mathcal{P}_{\sigma, \mathcal{N}}$ has the form in (3.4).

3.2 Step 2: Deducing a hypothesis for an explicit form for the Petz map, by considering Gaussian input states

In this step, we continue working with Ansatzes 1-3, with our main objective being to arrive at a hypothesis for the action of the Petz recovery map $\mathcal{P}_{\sigma_0, \mathcal{N}_0}$ on the mean vector and covariance matrix of an input Gaussian state. Here we consider the serial concatenation of the three completely positive maps in (3.11)–(3.13). We begin by considering the action of the last completely positive map on a zero-mean Gaussian input state ω_0 . To this end, recall from [BBP15, Appendix C] that if ω_0 and σ_0 are zero-mean Gaussian states, then $\sqrt{\sigma_0} \omega_0 \sqrt{\sigma_0}$ is an (unnormalized) Gaussian operator with zero mean vector and covariance matrix given by

$$V_{\sqrt{\sigma_0} \omega_0 \sqrt{\sigma_0}} = V_{\sigma_0} - \left(V_{\sqrt{\sigma_0}} - V_{\sigma_0} \right) (V_{\omega_0} + V_{\sigma_0})^{-1} \left(V_{\sqrt{\sigma_0}} - V_{\sigma_0} \right). \quad (3.20)$$

Applying a formula from [BBP15, Appendix B-2] (while noting our different convention for Gaussian states), we find that

$$V_{\sqrt{\sigma_0}} = \left(\sqrt{I + (V_{\sigma_0} \Omega)^{-2}} + I \right) V_{\sigma_0}, \quad (3.21)$$

which is a symmetric matrix because V_{σ_0} is. Indeed, consider that

$$V_{\sqrt{\sigma_0}}^T = \left[\left(\sqrt{I + (V_{\sigma_0} \Omega)^{-2}} + I \right) V_{\sigma_0} \right]^T = V_{\sigma_0} \left(\sqrt{I + (\Omega V_{\sigma_0})^{-2}} + I \right) \quad (3.22)$$

$$= \Omega^{-1} \Omega V_{\sigma_0} \left(\sqrt{I + (\Omega V_{\sigma_0})^{-2}} + I \right) = \Omega^{-1} \left(\sqrt{I + (\Omega V_{\sigma_0})^{-2}} + I \right) \Omega V_{\sigma_0} \quad (3.23)$$

$$= \left(\sqrt{\Omega^{-1} \left[I + (\Omega V_{\sigma_0})^{-2} \right] \Omega} + I \right) V_{\sigma_0} = \left(\sqrt{\left[I + (\Omega^{-1} \Omega V_{\sigma_0} \Omega)^{-2} \right]} + I \right) V_{\sigma_0} \quad (3.24)$$

$$= \left(\sqrt{I + (V_{\sigma_0} \Omega)^{-2}} + I \right) V_{\sigma_0} = V_{\sqrt{\sigma_0}}. \quad (3.25)$$

The equality in (3.21) implies that

$$V_{\sqrt{\sigma_0}} - V_{\sigma_0} = \sqrt{I + (V_{\sigma_0} \Omega)^{-2}} V_{\sigma_0}, \quad (3.26)$$

and in turn, after substituting into (3.20), that

$$V_{\sqrt{\sigma_0} \omega_0 \sqrt{\sigma_0}} = V_{\sigma_0} - \sqrt{I + (V_{\sigma_0} \Omega)^{-2}} V_{\sigma_0} (V_{\omega_0} + V_{\sigma_0})^{-1} V_{\sigma_0} \sqrt{I + (\Omega V_{\sigma_0})^{-2}}. \quad (3.27)$$

Thus, (3.27) establishes the action of the completely positive map $(\cdot) \rightarrow \sqrt{\sigma_0}(\cdot)\sqrt{\sigma_0}$ on an arbitrary zero-mean Gaussian state ω_0 .

From this discussion we already start seeing that the Petz map constructed out of a Gaussian state σ and a Gaussian channel \mathcal{N} should send normalized Gaussian states to normalized Gaussian states, because (i) conjugation by the square root of a Gaussian state (or the inverse square root of

a Gaussian state as we will see) preserves the Gaussian form; (ii) the adjoint of a Gaussian channel is still Gaussian; and (iii) the Petz map is a priori known to be trace-preserving whenever $\mathcal{N}(\sigma)$ is a faithful state [Pet86, Pet88, OP93]. Then, [PMGH15, Theorem III.1] ensures that \mathcal{P} must act as in (2.21), for some X_P , Y_P , and δ_P to be determined.

With this preliminary identity in hand, we are ready to determine a hypothesis for the explicit action of $\mathcal{P}_{\sigma_0, \mathcal{N}_0}$. For the sake of simplicity, we consider the input Gaussian state to have vanishing first moments. In any case, since we are working to deduce a hypothesis for an explicit form for the Petz map, this is by no means a loss of generality. By applying (3.27) and Ansatz 2 (that the following density operator transformation $\omega \rightarrow \omega^{-1}$ induces the transformation $V_\omega \rightarrow -V_\omega$ on the level of covariance matrices), we can conclude that the completely positive map in (3.11) has the following effect on covariance matrices:

$$\begin{aligned} & V_{\sqrt{\mathcal{N}_0(\sigma_0)}^{-1} \omega_0 \sqrt{\mathcal{N}_0(\sigma_0)}^{-1}} \\ &= -V_{\mathcal{N}(\sigma)} - \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}}. \end{aligned} \quad (3.28)$$

In the above, we have also used the identities $V_{\mathcal{N}_0(\sigma_0)} = V_{\mathcal{N}(\sigma)}$ and $V_{\omega_0} = V_\omega$. So now we consider further concatenating with the completely positive map in (3.12), by applying (2.25) and Ansatz 3 (that X is invertible):

$$\begin{aligned} & V_{\mathcal{N}_0^\dagger(\sqrt{\mathcal{N}_0(\sigma_0)}^{-1} \omega_0 \sqrt{\mathcal{N}_0(\sigma_0)}^{-1})} = \\ & X^{-1} \left[-V_{\mathcal{N}(\sigma)} - \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}} + Y \right] X^{-T}. \end{aligned} \quad (3.29)$$

But consider that $V_{\mathcal{N}(\sigma)} = X V_\sigma X^T + Y$, so that (3.29) simplifies as follows:

$$\begin{aligned} & V_{\mathcal{N}_0^\dagger(\sqrt{\mathcal{N}_0(\sigma_0)}^{-1} \omega_0 \sqrt{\mathcal{N}_0(\sigma_0)}^{-1})} \\ &= X^{-1} \left[- (X V_\sigma X^T + Y) - \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}} + Y \right] X^{-T} \\ &= X^{-1} \left[-X V_\sigma X^T - \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}} \right] X^{-T} \end{aligned} \quad (3.30)$$

$$= -V_\sigma - X^{-1} \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}} X^{-T}. \quad (3.31)$$

So then we can finally consider the serial concatenation of the three completely positive maps in (3.11)–(3.13):

$$\begin{aligned} & V_{\sqrt{\sigma_0} \mathcal{N}_0^\dagger(\sqrt{\mathcal{N}_0(\sigma_0)}^{-1} \omega_0 \sqrt{\mathcal{N}_0(\sigma_0)}^{-1}) \sqrt{\sigma_0}} \\ &= V_\sigma - \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \\ &\quad \times \left(-V_\sigma - X^{-1} \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}} X^{-T} + V_\sigma \right)^{-1} \\ &\quad \times V_\sigma \sqrt{I + (\Omega V_\sigma)^{-2}} \end{aligned} \quad (3.32)$$

$$\begin{aligned}
&= V_\sigma - \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \\
&\quad \times \left(-X^{-1} \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} (V_\omega - V_{\mathcal{N}(\sigma)})^{-1} V_{\mathcal{N}(\sigma)} \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}} X^{-T} \right)^{-1} \\
&\quad \times V_\sigma \sqrt{I + (\Omega V_\sigma)^{-2}}
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
&= V_\sigma + \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma X^T \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}}^{-1} V_{\mathcal{N}(\sigma)}^{-1} (V_\omega - V_{\mathcal{N}(\sigma)}) \\
&\quad \times V_{\mathcal{N}(\sigma)}^{-1} \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}}^{-1} X V_\sigma \sqrt{I + (V_\sigma \Omega)^{-2}}.
\end{aligned} \tag{3.34}$$

An inspection of (3.34) above suggests that the Petz map $\mathcal{P}_{\sigma_0, \mathcal{N}_0}$ is a quantum Gaussian channel with the following action on an input covariance matrix V_ω :

$$V_{\mathcal{P}_{\sigma_0, \mathcal{N}_0}(\omega_0)} = X_P V_\omega X_P^T + Y_P, \tag{3.35}$$

where

$$X_P \equiv \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma X^T \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}}^{-1} V_{\mathcal{N}(\sigma)}^{-1}, \tag{3.36}$$

$$Y_P \equiv V_\sigma - X_P V_{\mathcal{N}(\sigma)} X_P^T. \tag{3.37}$$

Combining with the development in Section 3.1, the results in (3.35), (3.19) and [PMGH15, Theorem III.1] imply that in general

$$\mathcal{P}_{\sigma, \mathcal{N}} : \begin{cases} V & \longmapsto X_P V X_P^T + Y_P \\ s & \longmapsto X_P s + \delta_P \end{cases}, \tag{3.38}$$

where

$$\delta_P \equiv s_\sigma - X_P (X s_\sigma + \delta), \tag{3.39}$$

and δ is the vector appearing in (2.21); it follows because

$$\mathcal{P}_{\sigma, \mathcal{N}}(\omega) = D_{s_\sigma}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0} \left(D_{X s_\sigma + \delta \omega} D_{X s_\sigma + \delta}^\dagger \right) D_{s_\sigma}, \tag{3.40}$$

which implies that

$$s_{\mathcal{P}_{\sigma, \mathcal{N}}(\omega)} = X_P (s_\omega - X s_\sigma - \delta) + s_\sigma. \tag{3.41}$$

So by using Ansatzes 1-3, we have arrived at our hypothesis (3.38) for the Gaussian form of the Petz map $\mathcal{P}_{\sigma, \mathcal{N}}$. In the next two sections, we give a detailed proof that the Gaussian channel specified in (3.38) is indeed equal to the Petz map $\mathcal{P}_{\sigma, \mathcal{N}}$.

3.3 Step 3: The Gaussian Petz map satisfies the Petz equations for all Hilbert–Schmidt operators

In this section, we prove that the hypothesis (3.38) for the Petz map satisfies the equations in (2.1) for all Hilbert–Schmidt operators. Recall that an operator T is Hilbert–Schmidt if

$$\|T\|_2 \equiv \sqrt{\text{Tr}[T^\dagger T]} < \infty. \tag{3.42}$$

Let T act on a tensor product of n separable Hilbert spaces (i.e., n modes). Its characteristic function is defined by

$$\chi_T(w) = \text{Tr}[T D_{-w}], \quad (3.43)$$

where $w \in \mathbb{R}^{2n}$. Thus, we can write T in terms of its characteristic function as

$$T = \int \frac{d^{2n}w}{(2\pi)^n} \chi_T(w) D_w. \quad (3.44)$$

Suppose that T_1 and T_2 are Hilbert–Schmidt operators. In order to demonstrate that our hypothesis (3.38) for $\mathcal{P}_{\sigma, \mathcal{N}}$ is in fact correct, we first show that the following equation is satisfied for this choice and for all Hilbert–Schmidt T_1 and T_2 :

$$\langle T_2, \mathcal{N}^\dagger(T_1) \rangle_\sigma = \langle \mathcal{P}_{\sigma, \mathcal{N}}^\dagger(T_2), T_1 \rangle_{\mathcal{N}(\sigma)}. \quad (3.45)$$

Using definitions and an expansion of T_1 and T_2 in terms of their characteristic functions $\chi_{T_1}(w_1)$ and $\chi_{T_2}(w_2)$, respectively, where $w_1, w_2 \in \mathbb{R}^{2n}$, we find that (3.45) is equivalent to

$$\begin{aligned} & \int \int \frac{d^{2n}w_1 d^{2n}w_2}{(2\pi)^{2n}} \chi_{T_2}^*(w_2) \chi_{T_1}(w_1) \text{Tr}[\sigma^{1/2} D_{-w_2} \sigma^{1/2} \mathcal{N}^\dagger(D_{w_1})] \\ &= \int \int \frac{d^{2n}w_1 d^{2n}w_2}{(2\pi)^{2n}} \chi_{T_2}^*(w_2) \chi_{T_1}(w_1) \text{Tr}[\mathcal{P}_{\sigma, \mathcal{N}}^\dagger(D_{-w_2}) \mathcal{N}(\sigma)^{1/2} D_{w_1} \mathcal{N}(\sigma)^{1/2}]. \end{aligned} \quad (3.46)$$

Thus, if we show that the following holds for all $w_1, w_2 \in \mathbb{R}^{2n}$

$$\text{Tr}[\sigma^{1/2} D_{-w_2} \sigma^{1/2} \mathcal{N}^\dagger(D_{w_1})] = \text{Tr}[\mathcal{P}_{\sigma, \mathcal{N}}^\dagger(D_{-w_2}) \mathcal{N}(\sigma)^{1/2} D_{w_1} \mathcal{N}(\sigma)^{1/2}], \quad (3.47)$$

then the statement in (3.45) is shown for all Hilbert–Schmidt operators. So we proceed with proving (3.47).

We first show that it suffices to verify (3.47) when σ is a zero-mean Gaussian state and \mathcal{N} is a zero-displacement Gaussian channel. Here we make use of (2.26), (2.27), and (2.29). Consider that

$$\sigma^{1/2} = D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma}, \quad (3.48)$$

$$\mathcal{N}(\sigma) = D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0) D_{X_{s_\sigma} + \delta}, \quad (3.49)$$

$$\mathcal{N}(\cdot) = D_\delta^\dagger \mathcal{N}_0(\cdot) D_\delta, \quad (3.50)$$

$$\mathcal{P}_{\sigma, \mathcal{N}}(\cdot) = D_{\delta_P}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0}(\cdot) D_{\delta_P}, \quad (3.51)$$

where δ_P is defined as in (3.4). We can then rewrite the left-hand side of (3.47) as

$$\begin{aligned} & \text{Tr}[\sigma^{1/2} D_{-w_2} \sigma^{1/2} \mathcal{N}^\dagger(D_{w_1})] \\ &= \text{Tr}[\mathcal{N}(\sigma^{1/2} D_{-w_2} \sigma^{1/2}) D_{w_1}] \end{aligned} \quad (3.52)$$

$$= \text{Tr}[D_\delta^\dagger \mathcal{N}_0(D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma} D_{-w_2} D_{s_\sigma}^\dagger \sigma_0^{1/2} D_{s_\sigma}) D_\delta D_{w_1}] \quad (3.53)$$

$$= \text{Tr}[D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0^{1/2} D_{s_\sigma} D_{-w_2} D_{s_\sigma}^\dagger \sigma_0^{1/2}) D_{X_{s_\sigma} + \delta} D_{w_1}] \quad (3.54)$$

$$= \text{Tr}[\mathcal{N}_0(\sigma_0^{1/2} D_{s_\sigma} D_{-w_2} D_{s_\sigma}^\dagger \sigma_0^{1/2}) D_{X_{s_\sigma} + \delta} D_{w_1} D_{X_{s_\sigma} + \delta}^\dagger] \quad (3.55)$$

$$= \exp(-i(X_{s_\sigma} + \delta)^T \Omega w_1 + i s_\sigma^T \Omega w_2) \text{Tr}[\mathcal{N}_0(\sigma_0^{1/2} D_{-w_2} \sigma_0^{1/2}) D_{w_1}] \quad (3.56)$$

$$= \exp(-i(X_{s_\sigma} + \delta)^T \Omega w_1 + i s_\sigma^T \Omega w_2) \text{Tr}[\sigma_0^{1/2} D_{-w_2} \sigma_0^{1/2} \mathcal{N}_0^\dagger(D_{w_1})]. \quad (3.57)$$

We can rewrite the right-hand side of (3.47) as

$$\begin{aligned} & \text{Tr}[\mathcal{P}_{\sigma, \mathcal{N}}^\dagger (D_{-w_2}) \mathcal{N}(\sigma)^{1/2} D_{w_1} \mathcal{N}(\sigma)^{1/2}] \\ &= \text{Tr}[D_{-w_2} \mathcal{P}_{\sigma, \mathcal{N}} (\mathcal{N}(\sigma)^{1/2} D_{w_1} \mathcal{N}(\sigma)^{1/2})] \end{aligned} \quad (3.58)$$

$$= \text{Tr}[D_{-w_2} D_{\delta_P}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0} (D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0)^{1/2} D_{X_{s_\sigma} + \delta} D_{w_1} D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0)^{1/2} D_{X_{s_\sigma} + \delta}) D_{\delta_P}] \quad (3.59)$$

$$= \text{Tr}[D_{-w_2} D_{\delta_P}^\dagger D_{X_P[X_{s_\sigma} + \delta]}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0} (\mathcal{N}_0(\sigma_0)^{1/2} D_{X_{s_\sigma} + \delta} D_{w_1} D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0)^{1/2}) D_{X_P[X_{s_\sigma} + \delta]} D_{\delta_P}]. \quad (3.60)$$

Considering that

$$D_{X_P[X_{s_\sigma} + \delta]} D_{\delta_P} = D_{\delta_P + X_P[X_{s_\sigma} + \delta]} e^{i\phi} = D_{s_\sigma} e^{i\phi}, \quad (3.61)$$

which follows from (3.4) and (2.8), we find that (3.60) is equal to

$$\begin{aligned} & \text{Tr}[D_{-w_2} D_{s_\sigma}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0} (\mathcal{N}_0(\sigma_0)^{1/2} D_{X_{s_\sigma} + \delta} D_{w_1} D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0)^{1/2}) D_{s_\sigma}] \\ &= \text{Tr}[D_{s_\sigma} D_{-w_2} D_{s_\sigma}^\dagger \mathcal{P}_{\sigma_0, \mathcal{N}_0} (\mathcal{N}_0(\sigma_0)^{1/2} D_{X_{s_\sigma} + \delta} D_{w_1} D_{X_{s_\sigma} + \delta}^\dagger \mathcal{N}_0(\sigma_0)^{1/2})] \end{aligned} \quad (3.62)$$

$$= \exp(-i(X_{s_\sigma} + \delta)^T \Omega w_1 + i s_\sigma^T \Omega w_2) \text{Tr}[D_{-w_2} \mathcal{P}_{\sigma_0, \mathcal{N}_0} (\mathcal{N}_0(\sigma_0)^{1/2} D_{w_1} \mathcal{N}_0(\sigma_0)^{1/2})] \quad (3.63)$$

$$= \exp(-i(X_{s_\sigma} + \delta)^T \Omega w_1 + i s_\sigma^T \Omega w_2) \text{Tr}[\mathcal{P}_{\sigma_0, \mathcal{N}_0}^\dagger (D_{-w_2}) \mathcal{N}_0(\sigma_0)^{1/2} D_{w_1} \mathcal{N}_0(\sigma_0)^{1/2}]. \quad (3.64)$$

Observe that the phases in (3.57) and (3.64) are equal. Thus, if the goal is to show the equality in (3.47), then our above development proves that it suffices to establish the following equality:

$$\text{Tr}[\sigma_0^{1/2} D_{-w_2} \sigma_0^{1/2} \mathcal{N}_0^\dagger(D_{w_1})] = \text{Tr}[\mathcal{P}_{\sigma_0, \mathcal{N}_0}^\dagger (D_{-w_2}) \mathcal{N}_0(\sigma_0)^{1/2} D_{w_1} \mathcal{N}_0(\sigma_0)^{1/2}]. \quad (3.65)$$

So now we focus on establishing (3.65).

To begin with, consider from (2.24) that

$$\mathcal{N}_0^\dagger(D_{w_1}) = D_{\Omega X^T \Omega^T w_1} \exp\left(-\frac{1}{4}(\Omega^T w_1)^T Y \Omega^T w_1\right). \quad (3.66)$$

Thus, the left-hand side of (3.65) reduces to

$$\text{Tr}[\sigma_0^{1/2} D_{-w_2} \sigma_0^{1/2} D_{\Omega X^T \Omega^T w_1}] \exp\left(-\frac{1}{4}(\Omega^T w_1)^T Y \Omega^T w_1\right). \quad (3.67)$$

Similarly, from (2.24) and (3.38), we have that

$$\mathcal{P}_{\sigma_0, \mathcal{N}_0}^\dagger(D_{-w_2}) = D_{-\Omega X_P^T \Omega^T w_2} \exp\left(-\frac{1}{4}(\Omega^T w_2)^T Y_P \Omega^T w_2\right), \quad (3.68)$$

so that the right-hand side of (3.65) reduces to

$$\text{Tr}[D_{-\Omega X_P^T \Omega^T w_2} \mathcal{N}_0(\sigma_0)^{1/2} D_{w_1} \mathcal{N}_0(\sigma_0)^{1/2}] \exp\left(-\frac{1}{4}(\Omega^T w_2)^T Y_P \Omega^T w_2\right). \quad (3.69)$$

So we should show the equality of (3.67) and (3.69), in order to establish the equality in (3.65).

To this end, Lemma 5 below is helpful for us. Invoking it, we find that the left-most expression in (3.67) reduces as

$$\begin{aligned}
& \text{Tr}[\sigma_0^{1/2} D_{-w_2} \sigma_0^{1/2} D_{\Omega X^T \Omega^T w_1}] \\
&= \exp \left(-\frac{1}{4} (\Omega X^T \Omega^T w_1)^T \Omega^T V_\sigma \Omega \Omega X^T \Omega^T w_1 - \frac{1}{4} w_2^T \Omega^T V_\sigma \Omega w_2 \right. \\
&\quad \left. + \frac{1}{2} (\Omega X^T \Omega^T w_1)^T \Omega^T \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \Omega w_2 \right) \\
&= \exp \left(-\frac{1}{4} (\Omega^T w_1)^T X V_\sigma X^T \Omega^T w_1 - \frac{1}{4} w_2^T \Omega^T V_\sigma \Omega w_2 - \frac{1}{2} (\Omega^T w_1)^T X \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \Omega w_2 \right).
\end{aligned} \tag{3.70}$$

$$\tag{3.71}$$

So this implies that (3.67) is equal to

$$\begin{aligned}
& \exp \left(-\frac{1}{4} (\Omega^T w_1)^T X V_\sigma X^T \Omega^T w_1 - \frac{1}{4} w_2^T \Omega^T V_\sigma \Omega w_2 \right. \\
&\quad \left. - \frac{1}{2} (\Omega^T w_1)^T X \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \Omega w_2 - \frac{1}{4} (\Omega^T w_1)^T Y \Omega^T w_1 \right) \\
&= \exp \left(-\frac{1}{4} (\Omega^T w_1)^T V_{\mathcal{N}(\sigma)} \Omega^T w_1 - \frac{1}{4} w_2^T \Omega^T V_\sigma \Omega w_2 - \frac{1}{2} (\Omega^T w_1)^T X \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \Omega w_2 \right).
\end{aligned} \tag{3.72}$$

Invoking Lemma 5 again, we find that the left-most expression in (3.69) reduces as

$$\begin{aligned}
& \text{Tr}[D_{-\Omega X_P^T \Omega^T w_2} \mathcal{N}_0(\sigma_0)^{1/2} D_{w_1} \mathcal{N}_0(\sigma_0)^{1/2}] \\
&= \exp \left(-\frac{1}{4} w_1^T \Omega^T V_{\mathcal{N}(\sigma)} \Omega w_1 - \frac{1}{4} (\Omega X_P^T \Omega^T w_2)^T \Omega^T V_{\mathcal{N}(\sigma)} \Omega \Omega X_P^T \Omega^T w_2 \right. \\
&\quad \left. + \frac{1}{2} w_1^T \Omega^T \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} \Omega \Omega X_P^T \Omega^T w_2 \right)
\end{aligned} \tag{3.73}$$

$$\begin{aligned}
&= \exp \left(-\frac{1}{4} w_1^T \Omega^T V_{\mathcal{N}(\sigma)} \Omega w_1 - \frac{1}{4} (\Omega^T w_2)^T X_P V_{\mathcal{N}(\sigma)} X_P^T \Omega^T w_2 \right. \\
&\quad \left. - \frac{1}{2} w_1^T \Omega^T \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}} V_{\mathcal{N}(\sigma)} X_P^T \Omega^T w_2 \right).
\end{aligned} \tag{3.74}$$

So this implies that (3.69) is equal to

$$\begin{aligned}
& \exp \left(-\frac{1}{4} w_1^T \Omega^T V_{\mathcal{N}(\sigma)} \Omega w_1 - \frac{1}{4} (\Omega^T w_2)^T X_P V_{\mathcal{N}(\sigma)} X_P^T \Omega^T w_2 \right. \\
& \quad \left. - \frac{1}{2} w_1^T \Omega^T \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2} V_{\mathcal{N}(\sigma)}} X_P^T \Omega^T w_2 - \frac{1}{4} (\Omega^T w_2)^T Y_P \Omega^T w_2 \right) \\
& = \exp \left(-\frac{1}{4} w_1^T \Omega^T V_{\mathcal{N}(\sigma)} \Omega w_1 - \frac{1}{4} (\Omega^T w_2)^T V_\sigma \Omega^T w_2 - \frac{1}{2} w_1^T \Omega^T \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2} V_{\mathcal{N}(\sigma)}} X_P^T \Omega^T w_2 \right). \tag{3.75}
\end{aligned}$$

Consider that

$$\begin{aligned}
& \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2} V_{\mathcal{N}(\sigma)}} X_P^T \\
& = \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2} V_{\mathcal{N}(\sigma)}} \left(\sqrt{I + (V_\sigma \Omega)^{-2} V_\sigma} X^T \sqrt{I + (\Omega V_{\mathcal{N}(\sigma)})^{-2}}^{-1} V_{\mathcal{N}(\sigma)}^{-1} \right)^T \tag{3.76}
\end{aligned}$$

$$= \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2} V_{\mathcal{N}(\sigma)}} V_{\mathcal{N}(\sigma)}^{-1} \sqrt{I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2}}^{-1} X \sqrt{I + (V_\sigma \Omega)^{-2} V_\sigma} \tag{3.77}$$

$$= X \sqrt{I + (V_\sigma \Omega)^{-2} V_\sigma}, \tag{3.78}$$

which finally implies that (3.75) is equal to

$$\begin{aligned}
& \exp \left(-\frac{1}{4} w_1^T \Omega^T V_{\mathcal{N}(\sigma)} \Omega w_1 - \frac{1}{4} (\Omega^T w_2)^T V_\sigma \Omega^T w_2 - \frac{1}{2} w_1^T \Omega^T X \sqrt{I + (V_\sigma \Omega)^{-2} V_\sigma} \Omega^T w_2 \right) \\
& = \exp \left(-\frac{1}{4} (\Omega^T w)^T V_{\mathcal{N}(\sigma)} \Omega^T w_1 - \frac{1}{4} w_2^T \Omega^T V_\sigma \Omega w_2 - \frac{1}{2} (\Omega^T w_1)^T X \sqrt{I + (V_\sigma \Omega)^{-2} V_\sigma} \Omega w_2 \right). \tag{3.79}
\end{aligned}$$

Comparing (3.79) with (3.72), we see that we have shown the equality in (3.65), which concludes the proof once Lemma 5 is established.

Before proving Lemma 5, we recall the following result. Although an analogous formula was already established in part by [BBP15, Appendix B], we provide a self-contained proof for the sake of completeness.

Lemma 4 (Square root of Gaussian states) *Let σ_0 be a Gaussian state with vanishing first moments, i.e. $s_{\sigma_0} = 0$. Then its uniquely defined square root $\sqrt{\sigma_0}$ is a trace class operator given by*

$$\sqrt{\sigma_0} = \left(\det V_{\sqrt{\sigma_0}} \right)^{1/4} \int \frac{d^{2n}w}{(2\pi)^n} e^{-\frac{1}{4} w^T V_{\sqrt{\sigma_0}} w} D_{\Omega w}, \tag{3.80}$$

where $V_{\sqrt{\sigma_0}}$ is given by (3.21).

Proof. Call K the right hand side of (3.80). Since the square root is uniquely defined, it suffices to show that $K^2 = \sigma_0$. In the following we will use the shorthand $U \equiv V_{\sqrt{\sigma_0}} > 0$, where the strict positivity can be readily verified using (3.21), and is also a consequence of U being a legitimate

quantum covariance matrix. We obtain

$$K^2 = \left((\det U)^{1/4} \int \frac{d^{2n}w}{(2\pi)^n} e^{-\frac{1}{4}w^T U w} D_{\Omega w} \right)^2 \quad (3.81)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}w d^{2n}z}{(2\pi)^{2n}} e^{-\frac{1}{4}w^T U w - \frac{1}{4}z^T U z} D_{\Omega w} D_{\Omega z} \quad (3.82)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}w d^{2n}z}{(2\pi)^{2n}} e^{-\frac{1}{4}w^T U w - \frac{1}{4}z^T U z - \frac{i}{2}w^T \Omega z} D_{\Omega(w+z)}. \quad (3.83)$$

Let us introduce the new variables $x \equiv w + z$ and $y \equiv \frac{w-z}{2}$, in terms of which we obtain

$$K^2 = (\det U)^{1/2} \int \frac{d^{2n}x d^{2n}y}{(2\pi)^{2n}} e^{-\frac{1}{8}x^T U x - \frac{1}{2}(y + \frac{i}{2}U^{-1}\Omega x)^T U (y + \frac{i}{2}U^{-1}\Omega x) - \frac{1}{8}x^T \Omega^T U^{-1}\Omega x} D_{\Omega x} \quad (3.84)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}x}{(2\pi)^n} e^{-\frac{1}{8}x^T (U + \Omega^T U^{-1}\Omega) x} \left(\int \frac{d^{2n}y}{(2\pi)^n} e^{-\frac{1}{2}(y + \frac{i}{2}U^{-1}\Omega x)^T U (y + \frac{i}{2}U^{-1}\Omega x)} \right) D_{\Omega x} \quad (3.85)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}x}{(2\pi)^n} e^{-\frac{1}{4}x^T V x} \left(\int \frac{d^{2n}\tilde{y}}{(2\pi)^n} e^{-\frac{1}{2}\tilde{y}^T U \tilde{y}} \right) D_{\Omega x} \quad (3.86)$$

$$= \int \frac{d^{2n}x}{(2\pi)^n} e^{-\frac{1}{4}x^T V x} D_{\Omega x} \quad (3.87)$$

$$= \sigma_0, \quad (3.88)$$

where we defined the shifted variable $\tilde{y} \equiv y + \frac{i}{2}U^{-1}\Omega x$ to perform the internal Gaussian integral and in the last step we appealed to the representation (2.19). Moreover, in the above calculation we observed that

$$\Omega^T U^{-1}\Omega = \Omega U^{-1}\Omega^T = (U\Omega)^{-1}\Omega \quad (3.89)$$

$$= \left(\sqrt{I + (V_\sigma\Omega)^{-2}} V_\sigma\Omega + V_\sigma\Omega \right)^{-1} \Omega \quad (3.90)$$

$$= \left(\sqrt{I + (V_\sigma\Omega)^{-2}} V_\sigma\Omega - V_\sigma\Omega \right) \Omega \quad (3.91)$$

$$= V_\sigma - \sqrt{I + (V_\sigma\Omega)^{-2}} V_\sigma \quad (3.92)$$

and hence

$$U + \Omega^T U^{-1}\Omega = 2V_\sigma. \quad (3.93)$$

This concludes the proof of Lemma 4. ■

Lemma 5 *Let σ_0 be a Gaussian state with vanishing first moments $s_{\sigma_0} = 0$. Then for all $x, y \in \mathbb{R}^{2n}$ we have*

$$\chi_{\sqrt{\sigma_0} D_x \sqrt{\sigma_0}}(y) = \text{Tr} [D_{-y} \sqrt{\sigma_0} D_x \sqrt{\sigma_0}] \quad (3.94)$$

$$= \exp \left(-\frac{1}{4}x^T \Omega^T V_\sigma \Omega x - \frac{1}{4}y^T \Omega^T V_\sigma \Omega y + \frac{1}{2}x^T \Omega^T \sqrt{I + (V_\sigma\Omega)^{-2}} V_\sigma \Omega y \right). \quad (3.95)$$

Proof. To perform the computation, we just need to employ: (i) the representation (3.80) for the square root of a Gaussian state with zero mean, (ii) the composition identity (2.8), (iii) the

orthogonality relation (2.9); and (iv) the standard formula for a Gaussian integral, i.e.

$$\int \frac{d^{2n}z}{(2\pi)^n} e^{-\frac{1}{2}z^T U z + \frac{1}{2}a^T U z} = \frac{e^{\frac{1}{8}a^T U a}}{\sqrt{\det U}}, \quad (3.96)$$

valid for $U > 0$. Defining again $U \equiv V\sqrt{\sigma_0}$, we obtain

$$\chi_{\sqrt{\sigma_0}D_x\sqrt{\sigma_0}}(y) = \text{Tr} [D_{-y}\sqrt{\sigma_0}D_x\sqrt{\sigma_0}] \quad (3.97)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}w d^{2n}z}{(2\pi)^{2n}} \exp\left(-\frac{1}{4}w^T U w - \frac{1}{4}z^T U z\right) \text{Tr} [D_{-y}D_{\Omega w}D_xD_{\Omega z}] \quad (3.98)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}w d^{2n}z}{(2\pi)^{2n}} \exp\left(-\frac{1}{4}w^T U w - \frac{1}{4}z^T U z + \frac{i}{2}x^T z - \frac{i}{2}y^T w\right) \\ \times \text{Tr} [D_{\Omega w-y}D_{\Omega z+x}] \quad (3.99)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}w d^{2n}z}{(2\pi)^{2n}} \exp\left(-\frac{1}{4}w^T U w - \frac{1}{4}z^T U z + \frac{i}{2}x^T z - \frac{i}{2}y^T w\right) \\ \times (2\pi)^n \delta(\Omega w - y + \Omega z + x) \quad (3.100)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}w d^{2n}z}{(2\pi)^{2n}} \exp\left(-\frac{1}{4}w^T U w - \frac{1}{4}z^T U z + \frac{i}{2}x^T z - \frac{i}{2}y^T w\right) \\ \times (2\pi)^n \delta(w - \Omega(x - y) + z) \quad (3.101)$$

$$= (\det U)^{1/2} \int \frac{d^{2n}z}{(2\pi)^n} \exp\left(-\frac{1}{4}(\Omega(x - y) - z)^T U (\Omega(x - y) - z) \right. \\ \left. - \frac{1}{4}z^T U z + \frac{i}{2}x^T z - \frac{i}{2}y^T (\Omega(x - y) - z)\right) \quad (3.102)$$

$$= (\det U)^{1/2} \exp\left(-\frac{1}{4}(x - y)^T \Omega^T U \Omega(x - y) + \frac{i}{2}x^T \Omega y\right) \\ \times \int \frac{d^{2n}z}{(2\pi)^n} \exp\left(-\frac{1}{4}z^T U z + \frac{1}{2}(\Omega(x - y) + iU^{-1}(x + y))^T U z\right) \quad (3.103)$$

$$= \exp\left(-\frac{1}{4}(x - y)^T \Omega^T U \Omega(x - y) + \frac{i}{2}x^T \Omega y\right) \\ \times \exp\left(\frac{1}{8}(\Omega(x - y) + iU^{-1}(x + y))^T U (\Omega(x - y) + iU^{-1}(x + y))\right) \quad (3.104)$$

$$= \exp\left(-\frac{1}{4}x^T \Omega^T \frac{U + \Omega U^{-1} \Omega^T}{2} \Omega x - \frac{1}{4}y^T \Omega^T \frac{U + \Omega U^{-1} \Omega^T}{2} \Omega y \right. \\ \left. + \frac{1}{2}x^T \Omega^T \frac{U - \Omega U^{-1} \Omega^T}{2} \Omega y\right) \quad (3.105)$$

$$= \exp\left(-\frac{1}{4}x^T \Omega^T V_\sigma \Omega x - \frac{1}{4}y^T \Omega^T V_\sigma \Omega y + \frac{1}{2}x^T \Omega^T \sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma \Omega y\right). \quad (3.106)$$

In the last step, we used (3.93) and the analogous relation $U - \Omega U^{-1} \Omega^T = 2\sqrt{I + (V_\sigma \Omega)^{-2}} V_\sigma$, deduced again with the help of (3.92). ■

3.4 Step 4: The Gaussian Petz map satisfies the Petz equations for all bounded operators

Throughout Section 3.3, we showed that the Petz equation in (2.1) is satisfied by the Gaussian channel in (3.38) for all Hilbert–Schmidt operators. In this section we complete the argument by showing that the same is true for all bounded operators A, B in (2.1). Thus, as a consequence of the development in this section, we can conclude from a result of [Pet86, Pet88, OP93] that the Gaussian channel in (3.38) is in fact the Petz map for σ and \mathcal{N} .

The argument given here is standard, but we provide it here for completeness. Proceeding, we have to show that the following Petz equation

$$\langle A, \mathcal{N}^\dagger(B) \rangle_\sigma = \langle \mathcal{P}^\dagger(A), B \rangle_{\mathcal{N}(\sigma)} \quad (3.107)$$

is satisfied for all bounded A, B , supposing that we can verify it only for a restricted class of A, B , for instance, those which are finite-rank (note that finite-rank operators are Hilbert–Schmidt). Recall that a sequence $(T_n)_{n \in \mathbb{N}}$ of operators on a Hilbert space \mathcal{H} is said to be weakly convergent to T , and we write $T_n \xrightarrow{w} T$, if

$$\lim_{n \rightarrow \infty} \langle \alpha, T_n \beta \rangle = \langle \alpha, T \beta \rangle \quad \forall \alpha, \beta \in \mathcal{H}. \quad (3.108)$$

We start by recalling the well-known fact that finite-rank operators are weakly dense in the set of bounded operators. It is straightforward to show this for all bounded A : one has $\Pi_n A \Pi_n \xrightarrow{w} A$, with Π_n denoting the projector onto the first n vectors of the canonical basis. Indeed, taking arbitrary vectors $\alpha, \beta \in \mathcal{H}$, we have that

$$|\langle \Pi_n \alpha, A \Pi_n \beta \rangle - \langle \alpha, A \beta \rangle| = |\langle \Pi_n \alpha - \alpha, A \beta \rangle + \langle \Pi_n \alpha, A(\Pi_n \beta - \beta) \rangle| \quad (3.109)$$

$$\leq |\langle \Pi_n \alpha - \alpha, A \beta \rangle| + |\langle \Pi_n \alpha, A(\Pi_n \beta - \beta) \rangle| \quad (3.110)$$

$$\leq \|A\|_\infty (\|\alpha\| \|\Pi_n \beta - \beta\| + \|\beta\| \|\Pi_n \alpha - \alpha\|) \xrightarrow{n \rightarrow \infty} 0. \quad (3.111)$$

An important tool in our discussion will be the *uniform boundedness principle* [Bou87], which states that if a sequence of operators $(T_n)_{n \in \mathbb{N}}$ is such that the sequence of norms $(\|T_n \alpha\|)_{n \in \mathbb{N}}$ is bounded for all $\alpha \in \mathcal{H}$, then the sequence of operator norms $\|T_n\|_\infty$ is itself bounded.

Lemma 6 *Let $(A_n)_{n \in \mathbb{N}}$ be a weakly convergent sequence of operators. Then the sequence of operator norms $(\|A_n\|_\infty)_{n \in \mathbb{N}}$ is bounded.*

Proof. Pick an arbitrary $\alpha \in \mathcal{H}$, and consider the sequence of functionals $f_n^{(\alpha)} : \mathcal{H} \rightarrow \mathbb{C}$ acting as $f_n^{(\alpha)}(\beta) = \langle A_n \alpha, \beta \rangle$. Since $(A_n)_{n \in \mathbb{N}}$ is weakly convergent, $f_n^{(\alpha)}(\beta)$ has a limit in \mathbb{C} , and in particular it is bounded. Since this holds for all β , the uniform boundedness principle states that the norms $\|f_n^{(\alpha)}\|_\infty = \|A_n \alpha\|$ must be bounded as well. Since this holds for an arbitrary α , another application of the uniform boundedness principle guarantees that also $(\|A_n\|_\infty)_{n \in \mathbb{N}}$ is bounded. ■

Now we discuss some alternative definitions of weak convergence.

Lemma 7 *Given a sequence $(A_n)_{n \in \mathbb{N}}$ of operators on a Hilbert space, the following are equivalent:*

1. $A_n \xrightarrow{w} A$;

2. $\text{Tr}[\rho A_n] \rightarrow \text{Tr}[\rho A]$ for all states ρ ;
3. $\text{Tr}[Z A_n] \rightarrow \text{Tr}[Z A]$ for all trace-class Z .

Proof.

1. \Rightarrow 2. Since $A_n - A \xrightarrow{w} 0$, Lemma 6 ensures that there is a constant M such that for sufficiently large n $\|A_n - A\|_\infty \leq M < \infty$. Since ρ is a state, for all $\varepsilon > 0$ we can fix a projector Π onto a finite-dimensional subspace such that $\|\rho - \Pi\rho\Pi\|_1 \leq \frac{\varepsilon}{2M}$. Moreover, the weak convergence of A_n and the fact that $\Pi\rho\Pi$ has finite support imply that $\text{Tr}[\Pi\rho\Pi(A_n - A)] < \frac{\varepsilon}{2}$ for sufficiently large n . Then

$$|\text{Tr}[\rho(A_n - A)]| \leq |\text{Tr}[\Pi\rho\Pi(A_n - A)]| + |\text{Tr}[(\rho - \Pi\rho\Pi)(A_n - A)]| \quad (3.112)$$

$$\leq \frac{\varepsilon}{2} + \|\rho - \Pi\rho\Pi\|_1 \|A_n - A\|_\infty \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} M = \varepsilon, \quad (3.113)$$

for sufficiently large n . This shows that $\text{Tr}[\rho A_n] \rightarrow \text{Tr}[\rho A]$.

2. \Rightarrow 3. This follows directly because all trace-class operators can be written as a complex linear combination of four states.
3. \Rightarrow 1. This implication becomes clear once we choose Z to be the rank-one operator $Zx \equiv (\alpha, x)\beta$ and apply the definition of weak convergence (3.108).

This concludes the proof. ■

Corollary 8 *Let \mathcal{N} be a quantum channel. If a sequence of bounded operators A_n satisfies $A_n \xrightarrow{w} A$, then $\mathcal{N}^\dagger(A_n) \xrightarrow{w} \mathcal{N}^\dagger(A)$.*

Proof. We verify condition 2 of Lemma 7. Pick a state ρ . One has

$$\text{Tr}[\rho \mathcal{N}^\dagger(A_n)] = \text{Tr}[\mathcal{N}(\rho) A_n] \rightarrow \text{Tr}[\mathcal{N}(\rho) A] = \text{Tr}[\rho \mathcal{N}^\dagger(A)], \quad (3.114)$$

where we used again condition 2 of Lemma 7 in order to take the limit. ■

Now we come to our decisive tool:

Corollary 9 *Let $A_n \xrightarrow{w} A$ be a weakly convergent sequence of operators. Then the following holds for an arbitrary state ω and bounded operator B :*

$$\langle A_n, B \rangle_\omega \xrightarrow{n \rightarrow \infty} \langle A, B \rangle_\omega. \quad (3.115)$$

Proof. It suffices to note that $\omega^{1/2} B \omega^{1/2}$ is a trace-class operator. Applying condition 3 of Lemma 7 yields the statement. ■

Theorem 10 *If the Petz equation in (3.107) is satisfied for all finite-rank operators A, B , then the same is true for all bounded A, B .*

Proof. For bounded A, B , consider sequences of finite-rank operators $(A_n)_{n \in \mathbb{N}}$ and $(B_m)_{m \in \mathbb{N}}$ such that $A_n \xrightarrow{w} A$ and $B_m \xrightarrow{w} B$. Then we have

$$\langle A_n, \mathcal{N}^\dagger(B_m) \rangle_\sigma = \langle \mathcal{P}^\dagger(A_n), B_m \rangle_{\mathcal{N}(\sigma)}. \quad (3.116)$$

Since Corollary 8 implies that $\mathcal{N}^\dagger(B_m) \xrightarrow{w} \mathcal{N}^\dagger(B)$, we can safely use Corollary 9 to take the limit $m \rightarrow \infty$ on both sides, which yields

$$\langle A_n, \mathcal{N}^\dagger(B) \rangle_\sigma = \langle \mathcal{P}^\dagger(A_n), B \rangle_{\mathcal{N}(\sigma)}. \quad (3.117)$$

With the same argument we can now take the limit $n \rightarrow \infty$, and this concludes the proof. ■

4 Conclusion

The main result of this paper is Theorem 1, which establishes an explicit form for the Petz map as a bosonic Gaussian channel whenever the state σ and the channel \mathcal{N} are bosonic Gaussian. Our proof approach was first to consider three ansatzes in order to arrive at a hypothesis for the Gaussian form of the Petz map. These ansatzes included 1) working with the form of the Petz map in (2.3) in spite of the fact that $[\mathcal{N}(\sigma)]^{-1}$ is an unbounded operator, 2) negating the covariance matrix of the Gaussian state σ if σ is inverted, and 3) assuming that the X matrix in (2.20), corresponding to a Gaussian channel, is invertible. After deducing a hypothesis for an explicit form, we proved that this hypothesis is in fact correct, by demonstrating that the Gaussian Petz channel satisfies the equations in (3.107) for all bounded operators A and B . Additionally, our Appendix A, building on [BB69, Equation (30)], offers a powerful tool for computing products of exponentials of inhomogeneous quadratic Hamiltonians. We suspect that the ideas and tools presented in this paper will be useful for making future progress in Gaussian quantum information.

In future work, it would be interesting to determine whether the following inequality, considered in [BSW15a, SBW15], could be satisfied whenever all of the objects involved are Gaussian:

$$D(\rho \parallel \sigma) \geq D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)) - \log F(\rho, (\mathcal{P}_{\sigma, \mathcal{N}} \circ \mathcal{N})(\rho)). \quad (4.1)$$

More generally, one could consider the various inequalities proposed in [BSW15b] for the Gaussian case.

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A Golden rule to handle exponentials of inhomogeneous quadratic Hamiltonians

Very often in quantum optics one has to manipulate products of exponentials of (inhomogeneous) quadratic Hamiltonians, i.e., operators of the following form:

$$\mathcal{H} = \frac{i}{2}r^T\Omega Xr + is^T\Omega r + \frac{i}{2}a. \quad (\text{A.1})$$

For instance, a typical task consists in turning such a product into a single exponential of another quadratic operator of the same form. In the above equation, $r = (x_1, \dots, x_n, p_1, \dots, p_n)^T$ denotes the column vector of canonical coordinates, $[r, r^T] = i\Omega$ and ΩX can be assumed to be symmetric. Within the context of quantum optics, several methods have been developed to deal with such calculations, which can be very involved otherwise. In particular, a general formula for converting product of exponentials of quadratic operators into a single exponential has been found in [BB69, Equation (30)].

The main idea behind the approach we discuss here is not particularly novel and has been already successfully exploited in quantum optics. For a thorough review with many examples, we refer the reader to [Pur01, Chapter 2]. However, the particular example we present does not seem to have been considered before, and we believe it is of practical importance to make the kind of computations we performed in this paper much easier and more intuitive. To demonstrate the convenience of our method, we conclude this appendix with an alternative proof of Lemma 5.

The starting point is the observation that quadratic operators of the form (A.1) form a Lie algebra, a fact which is easily seen to be a consequence of the canonical commutation relations (2.5). Namely, it is easy to see that

$$\left[\frac{i}{2}r^T\Omega Xr + is^T\Omega r + \frac{i}{2}a, \frac{i}{2}r^T\Omega Yr + it^T\Omega r + \frac{i}{2}b \right] = \frac{i}{2}r^T\Omega[X, Y]r + i(Xt - Ys)^T\Omega r - is^T\Omega t. \quad (\text{A.2})$$

As is well-known, given $\mathcal{H}_1, \mathcal{H}_2$ of the form (A.1), the operator \mathcal{H}_3 satisfying

$$e^{\mathcal{H}_1}e^{\mathcal{H}_2} = e^{\mathcal{H}_3} \quad (\text{A.3})$$

depends only on the Lie algebra generated by \mathcal{H}_1 and \mathcal{H}_2 . Therefore, if we could construct an isomorphism turning the Lie algebra of quadratic Hamiltonians into a (low-dimensional) matrix algebra, we would be able to compute \mathcal{H}_3 as follows:

- (i) associate matrices M_1, M_2 to $\mathcal{H}_1, \mathcal{H}_2$ through the above isomorphism;
- (ii) compute the Lie algebra element M_3 such that $e^{M_1}e^{M_2} = e^{M_3}$;
- (iii) use one last time the isomorphism to translate M_3 back to a quadratic Hamiltonian \mathcal{H}_3 .

It turns out that such an isomorphism can be found. An explicit example is as follows:

$$\frac{i}{2}r^T\Omega Xr + is^T\Omega r + \frac{i}{2}a \quad \longleftrightarrow \quad \begin{pmatrix} 0 & s^T\Omega^T & a \\ 0 & X & s \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.4})$$

The matrix Lie algebra we will be concerned about is thus formed by matrices of the above form, with the only restriction that ΩX is symmetric. As expected, the commutator between two such matrices takes the form

$$\left[\begin{pmatrix} 0 & s^T \Omega^T & a \\ 0 & X & s \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^T \Omega^T & b \\ 0 & Y & s \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & (Xt - Ys)^T \Omega^T & -2s^T \Omega t \\ 0 & [X, Y] & Xt - Ys \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.5})$$

mimicking (A.2). In order to apply our strategy, we need to compute the exponential of a matrix belonging to our Lie algebra. It is an elementary exercise to show that

$$\exp \left[\begin{pmatrix} 0 & s^T \Omega^T & a \\ 0 & X & s \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & \left(\frac{I - e^{-X}}{X} s \right)^T \Omega^T & a + s^T \Omega \frac{X - \sinh X}{X^2} s \\ 0 & e^X & \frac{e^X - I}{X} s \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.6})$$

We conclude this appendix by presenting an alternative and perhaps more intuitive derivation of Lemma 5 that makes use of the Lie algebra isomorphism (A.4). The advantage of this proof is basically that it turns the cumbersome sequence of Gaussian integrals we performed in the main body into a sequence of 3×3 block-matrix multiplications.

Alternative proof of Lemma 5. As a preliminary step, we deduce from (2.15) the expression for $\sinh\left(\frac{i\Omega H_\sigma}{2}\right)$. Since $i\Omega H_\sigma$ has real eigenvalues, we can apply the identity $\sinh x = \frac{\coth(x)^{-1}}{\sqrt{1 - \coth(x)^2}}$ (valid for real x) and (2.15) to obtain

$$\sinh\left(\frac{i\Omega H_\sigma}{2}\right) = \frac{(iV\Omega)^{-1}}{\sqrt{I + (V\Omega)^{-2}}}. \quad (\text{A.7})$$

Now, let us show how to compute $\sqrt{\sigma_0} D_x \sqrt{\sigma_0}$ for any given x . We can employ the exponential form of σ_0 as given in (2.14), which in our case becomes $\sigma_0 = Z_\sigma e^{-\frac{1}{2} r^T H_\sigma r}$. For the sake of simplicity, we ignore the normalisation constant Z_σ for the moment. Also, let us omit the subscripts σ throughout the calculation. We find

$$\sqrt{\sigma_0} D_x \sqrt{\sigma_0} \propto e^{\frac{i}{4} r^T \Omega (-i\Omega H) r} e^{ix^T \Omega r} e^{\frac{i}{4} r^T \Omega (-i\Omega H) r} \quad (\text{A.8})$$

$$\stackrel{(i)}{\rightarrow} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\Omega H/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & x^T \Omega^T & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\Omega H/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{A.9})$$

$$\stackrel{(ii)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Omega H/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x^T \Omega^T & 0 \\ 0 & I & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Omega H/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A.10})$$

$$= \begin{pmatrix} 1 & x^T \Omega^T e^{-i\Omega H/2} & 0 \\ 0 & e^{-i\Omega H} & e^{-i\Omega H/2} x \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A.11})$$

$$\begin{aligned}
& \stackrel{\text{(iii)}}{=} \begin{pmatrix} 1 & \left(\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x \right)^T \Omega^T & 0 \\ 0 & I & \frac{1}{2 \sinh(\frac{i\Omega H}{2})} x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{4} x^T \Omega \frac{e^{-i\Omega H}}{\sinh(\frac{i\Omega H}{2})^2} x \\ 0 & e^{-i\Omega H} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
& \quad \times \begin{pmatrix} 1 & \left(-\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x \right)^T \Omega^T & 0 \\ 0 & I & -\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x \\ 0 & 0 & 1 \end{pmatrix} \tag{A.12}
\end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{(iv)}}{\longrightarrow} \exp \left(i \begin{pmatrix} 1 \\ \frac{1}{2 \sinh(\frac{i\Omega H}{2})} x \end{pmatrix}^T \Omega r \right) \exp \left(\frac{i}{2} r^T \Omega (-i\Omega H) r + \frac{i}{8} x^T \Omega \frac{e^{-i\Omega H}}{\sinh(\frac{i\Omega H}{2})^2} x \right) \\
& \quad \times \exp \left(i \begin{pmatrix} 1 \\ -\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x \end{pmatrix}^T \Omega r \right) \tag{A.13}
\end{aligned}$$

$$= \exp \left(\frac{i}{8} x^T \Omega \frac{e^{-i\Omega H}}{\sinh(\frac{i\Omega H}{2})^2} x \right) D_{\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x} e^{-\frac{1}{2} r^T H r} D_{-\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x} \tag{A.14}$$

$$\stackrel{\text{(v)}}{=} \exp \left(-\frac{i}{8} x^T \Omega \frac{\sinh(i\Omega H)}{\sinh(\frac{i\Omega H}{2})^2} x \right) D_{\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x} e^{-\frac{1}{2} r^T H r} D_{-\frac{1}{2 \sinh(\frac{i\Omega H}{2})} x} \tag{A.15}$$

$$\stackrel{\text{(vi)}}{=} \exp \left(-\frac{1}{4} x^T \Omega^T V \Omega x \right) D_{\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x} e^{-\frac{1}{2} r^T H r} D_{-\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x}. \tag{A.16}$$

The justification of these steps is as follows: (i) forward application of the isomorphism (A.4); (ii) exponential formula (A.6); (iii) direct verification; (iv) backward application of the isomorphism (A.4); (v) we use $x^T A x = \frac{1}{2} x^T (A + A^T) x$ to symmetrise the matrix inside the first exponential; (vi) we employ (A.7), the hyperbolic trigonometric identity $\frac{\sinh y}{\sinh(y/2)^2} = 2 \coth(y/2)$ and (2.15). Once we reintroduce the normalisation Z_σ , the above calculation shows that

$$\sqrt{\sigma_0} D_x \sqrt{\sigma_0} = e^{-\frac{1}{4} x^T \Omega^T V \Omega x} D_{\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x} \sigma_0 D_{-\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x}. \tag{A.17}$$

Then, using (2.8) and (2.17) we see that

$$\chi_{\sqrt{\sigma_0} D_x \sqrt{\sigma_0}}(y) = \text{Tr}[D_{-y} \sqrt{\sigma_0} D_x \sqrt{\sigma_0}] \tag{A.18}$$

$$= e^{-\frac{1}{4} x^T \Omega^T V \Omega x} \text{Tr} \left[D_{-y} D_{\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x} \sigma_0 D_{-\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x} \right] \tag{A.19}$$

$$= e^{-\frac{1}{4} x^T \Omega^T V \Omega x} \text{Tr} \left[D_{-\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x} D_{-y} D_{\frac{i}{2} \sqrt{I+(V\Omega)^{-2}} V \Omega x} \sigma_0 \right] \tag{A.20}$$

$$= e^{-\frac{1}{4} x^T \Omega^T V \Omega x} e^{\frac{1}{2} x^T \Omega^T \sqrt{I+(V\Omega)^{-2}} V \Omega y} \text{Tr} [D_{-y} \sigma_0] \tag{A.21}$$

$$= e^{-\frac{1}{4} x^T \Omega^T V \Omega x} e^{\frac{1}{2} x^T \Omega^T \sqrt{I+(V\Omega)^{-2}} V \Omega y} \chi_{\sigma_0}(y) \tag{A.22}$$

$$= \exp \left(-\frac{1}{4} x^T \Omega^T V \Omega x + \frac{1}{2} x^T \Omega^T \sqrt{I+(V\Omega)^{-2}} V \Omega y - \frac{1}{4} y^T \Omega^T V \Omega y \right), \tag{A.23}$$

which concludes the proof. ■

B Verifying complete positivity of the Gaussian Petz channel

One might want to verify explicitly the complete positivity condition for the Petz map stated in Theorem 1, even if we know from [Pet86, Pet88, OP93] that (2.3) has to be completely positive by construction. Recall that a Gaussian channel defined by (2.21) is completely positive if and only if [ARL14, Ser17]

$$Y + i\Omega - iX\Omega X^T \geq 0. \quad (\text{B.1})$$

We start with the following lemma:

Lemma 11 *For all V such that $V + i\Omega > 0$, the following identity holds*

$$(I + (\Omega V^{-2})^{-1/2} V^{-1} (V + i\Omega) V^{-1} (I + (V\Omega)^{-2})^{-1/2} = \frac{1}{V - i\Omega}. \quad (\text{B.2})$$

Proof. This is a straightforward calculation after decomposing V in the Williamson form as $V = S(D \oplus D)S^T$, where S is a symplectic matrix satisfying $S\Omega S^T = \Omega$ and D is a diagonal matrix of symplectic eigenvalues (note that all entries of D are larger than or equal to one). ■

With the above result in hand, we can write

$$\begin{aligned} Y_P + i\Omega - iX_P \Omega X_P^T &= V_\sigma + i\Omega - X_P (V_{\mathcal{N}(\sigma)} + i\Omega) X_P^T \\ &= (I + (V_\sigma \Omega)^{-2})^{1/2} V_\sigma \frac{1}{V_\sigma - i\Omega} V_\sigma (I + (\Omega V)^{-2})^{1/2} \\ &\quad - (I + (V_\sigma \Omega)^{-2})^{1/2} V_\sigma X^T \left(I + (\Omega V_{\mathcal{N}(\sigma)})^{-2} \right)^{-1/2} V_{\mathcal{N}(\sigma)}^{-1} (V_{\mathcal{N}(\sigma)} + i\Omega) \\ &\quad \times V_{\mathcal{N}(\sigma)}^{-1} \left(I + (V_{\mathcal{N}(\sigma)} \Omega)^{-2} \right)^{-1/2} X V_\sigma \left(I + (\Omega V_\sigma)^{-2} \right)^{1/2} \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} &= (I + (V_\sigma \Omega)^{-2})^{1/2} V_\sigma \frac{1}{V_\sigma - i\Omega} V_\sigma (I + (\Omega V)^{-2})^{1/2} \\ &\quad - (I + (V_\sigma \Omega)^{-2})^{1/2} V_\sigma X^T \frac{1}{V_{\mathcal{N}(\sigma)} - i\Omega} X V_\sigma \left(I + (\Omega V_\sigma)^{-2} \right)^{1/2} \end{aligned} \quad (\text{B.4})$$

$$= (I + (V_\sigma \Omega)^{-2})^{1/2} V_\sigma \left(\frac{1}{V_\sigma - i\Omega} - X^T \frac{1}{V_{\mathcal{N}(\sigma)} - i\Omega} X \right) V_\sigma \left(I + (\Omega V_\sigma)^{-2} \right)^{1/2}. \quad (\text{B.5})$$

Now, from $V_{\mathcal{N}(\sigma)} - i\Omega = X V_\sigma X^T + Y - i\Omega \geq X(V_\sigma - i\Omega)X^T$ we obtain

$$\frac{1}{V_\sigma - i\Omega} - X^T \frac{1}{V_{\mathcal{N}(\sigma)} - i\Omega} X \geq \frac{1}{V_\sigma - i\Omega} - X^T \frac{1}{X(V_\sigma - i\Omega)X^T} X \geq 0, \quad (\text{B.6})$$

as it follows from the inequality $A^{-1} \geq X^T (X A X^T)^{-1} X$, which is in turn valid for all invertible A and all matrices X with no more rows than columns and maximum rank. Plugging (B.6) into (B.5), we conclude the condition in (B.1) for X_P and Y_P , as desired.

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